

# On conformally invariant differential operators.

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## Abstract

We construct new families of conformally invariant differential operators acting on densities. We introduce a simple, direct approach which shows that all such operators arise via this construction when the degree is bounded by the dimension. The method relies on a study of well-known transformation laws and on Weyl's theory regarding identities holding "formally" vs. "by substitution". We also illustrate how this new method can strengthen existing results in the parabolic invariant theory for conformal geometries.

## 1 Introduction

This paper presents a construction of new families of conformally invariant differential operators acting on densities and partially shows that all such invariant operators arise via this construction. This project thus fits into the program of identifying invariants of parabolic geometries, a problem on which there is a rich literature and for which an invariant theory has been developed (see [8], [6],[16], [17] and references therein). In this paper we restrict attention to conformal geometry (and not CR or projective geometry).

The broad challenge of constructing local objects (scalars, tensors, differential operators etc) which exhibit a form of invariance under conformal changes of the underlying metric has been pursued for some time, partly in connection with questions in general relativity, see [23], [22], [21]. A number of closely related techniques for constructing such objects have been developed, e.g. the works of T.Y. Thomas [24], the Cartan conformal connection (see [20]), the Fefferman-Graham *ambient metric* [11], and the tractor calculus [5]. Our construction uses the ambient metric.

The method we employ for the construction of the new operators is the standard method with which one obtains conformally invariant scalar quantities using the ambient metric: Very roughly, the ambient metric provides a way to embed an entire conformal class  $(M, [g])$  (and also a conformal density defined over it) into an ambient pseudo-Riemannian manifold, so that any intrinsic scalar object one constructs in the ambient manifold will automatically be a conformal invariant of the original conformal class  $(M, [g])$ . Now, once one constructs conformally invariant objects (differential operators in this case), it

is a natural question to ask whether one has found all such objects which exhibit the required invariance. This can be thought of as a completeness question.

Our proof that all operators with the required conformal invariance arise via our construction (theorems 1, 2, 3) is direct and in a sense elementary. It presupposes no knowledge of representation theory—in particular no prior knowledge of [8], [6] is needed. Essentially, our proof relies on a careful study of the transformation laws under conformal re-scaling of the objects under consideration and the classical formalism of Weyl [25]. Our method has one limitation: It can only be applied if the degree  $d$  (see Definition 6) is less than or equal to the dimension  $n$ . For completeness, we also illustrate how the invariant theory developed in [6] can be applied to settle the case where the degree is greater than the dimension, provided  $n$  is odd and the densities satisfy certain additional restrictions (see theorem 4 below).

It is worth noting that a forthcoming paper by Hirachi (see [18]) develops an analogous direct argument to construct and prove completeness for CR-invariant operators, with applications to the asymptotic expansion of the Szegő kernel of strictly pseudo-convex domains. It is also interesting to note that our approach is in some sense analogous to the “direct method” that was used in Proposition 3.2 in [6] for the case of invariants of densities with degree bounded by the dimension. The authors in [6] used this method in the case of invariants involving only densities, not curvature tensors (the latter case is more subtle due to the algebraic complexity of the curvature and its covariant derivatives).

*Outline of the paper:* In section 2 we give a rigorous definition of conformal invariance for differential operators in curved space, and we recall the earlier known examples of such operators. In section 4 we recall the Fefferman-Graham ambient metric construction and explain in detail the sense in which this is a conformally invariant construction. We then construct the new operators and discuss some of their features (they arise in families and most of them vanish in flat space). In section 5 we state the completeness results, which we then prove in sections 6 and 7. In section 6 we first give a one-page synopsis of the argument, and then explain how the proof is naturally divided into three steps; we prove the first two (which are shorter) in the rest of section 6 and the lengthier one in section 7. Finally, in section 8 we present a straightforward adaptation of the methods in [6] to prove a completeness result in the case where the degree is higher than the dimension (in which case our new approach does not go through).

*History of the problem and strengthening of existing results:* The first pioneering construction of conformally invariant scalars (depending only on curvature terms) was carried out in [11], where Fefferman and Graham posed the geometric problem whether all invariants arose via their construction. The natural notion of conformal invariance for general curved structures is the one discussed in Definition 4 below. We will be using this notion of invariance throughout our paper.

The papers [8], [6] sought to address the problem posed in [11]. The paper [6]

of Bailey, Eastwood and Graham was the first to attack the geometric problem where (subject to certain restrictions in even dimensions) the authors proved completeness for conformal invariants which locally depend on the curvature and its derivatives. In [8] and also in [6] the authors also discuss the problem of determining all conformal invariants that depend on the derivatives of a single function (and these are called conformal invariants of densities). This latter question is studied as an interesting model problem, which does not apply directly to the geometric problem, see [13].

Now, the problem that we address in this paper is to construct and prove completeness for invariant *differential operators* (which locally depend on *both* curvature terms and derivatives of a function). Nonetheless, our approach to this problem succeeds in settling some cases that [8] and [6] left open:

In even dimensions, the authors in [6] solve the completeness problem for conformally invariant scalars provided the degree is strictly less than  $n$  ( $n$  being the dimension). Our method also captures the case where the degree is  $n$ .

Regarding conformal invariants of densities, the authors in [8] and [6] consider the problem in the setting of Euclidean space, where they require invariance of the local objects under the action of the conformal group. This is different from our setting, where we consider objects defined on all curved structures and the notion of conformal invariance we require is the one of definition 4. Nonetheless, in the setting of Riemannian operators which are conformally invariant in the sense of definition 4 (which is a stronger requirement than the invariance under the conformal group imposed in [8]), we can strengthen some of the results in [8], [6] regarding invariants of densities. See the discussion in section 5.

*Applications:* Fefferman and Hirachi [12] have proven that each new operator  $P_g$  governs the transformation law (under conformal re-scalings) of a certain scalar  $Q^{P_g}$  which depends on both the Weyl and the Ricci curvatures. Thus, the study of these new scalars can be seen as the study of the interplay between the Weyl and Ricci curvatures in a conformal class.

## 2 Formulation of the problem.

Our goal here is to explicitly construct all the *intrinsic* differential operators  $L_g(f)$ , defined on smooth Riemannian manifolds  $(M, g)$  of a fixed dimension  $n$ , that remain invariant under conformal re-scalings of the metric  $g$ . We first define *intrinsic* operators<sup>1</sup>.

**Definition 1** *An intrinsic differential operator  $L_{(M,g)}$  acting on scalar functions  $f \in C^\infty(M)$  is a differential operator associated to each Riemannian manifold  $(M, g)$  with the following properties:*

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<sup>1</sup>Our notion of intrinsicness in definition 1 requires the symbol of the operator to remain invariant under both orientation-preserving and orientation-reversing isometries. Thus, we are actually restricting attention to *even* invariants, in the language of [6]

1. There is a fixed polynomial expression  $P$  in the variables  $\frac{\partial^a g_{ij}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, (\det g_{ij})^{-1}$  and  $\frac{\partial^b f}{\partial x_1^{l_1} \dots \partial x_n^{l_n}}$  so that for any Riemannian manifold  $(M, g)$  of dimension  $n$  and any  $f \in C^\infty(M)$  we have in local coordinates the formula

$$L_{(M,g)}f = P\left(\frac{\partial^a g_{ij}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, (\det g_{ij})^{-1}, \frac{\partial^b f}{\partial x_1^{l_1} \dots \partial x_n^{l_n}}\right)$$

2. For any two Riemannian manifolds  $(M, g), (M', g')$ , isometric via the map  $\phi : M \rightarrow M'$ , and any  $f \in C^\infty(M)$ , we have that:  $L_{(M',g')}(\phi_*f) = \phi_*(L_{(M,g)}f)$ .

It is a classical result of Weyl [25] (see also [9]) that any intrinsic (also called Riemannian) differential operator can be written as a linear combination of complete contractions involving intrinsic “building blocks”:

Let  $R$  stand for the curvature tensor of the metric  $g$  (we are suppressing the lower indices of the curvature tensor). Let  $\nabla$  stand for the Levi-Civita connection of  $g$ . Weyl’s theory then ensures that for every *intrinsic* operator  $L_g(f)$ , there exists a fixed linear combination  $\sum_{x=1}^V a_x C_g^x(f)$  of complete contractions in the form:

$$\text{contr}(\nabla_{a_1 \dots a_{m_1}}^{(m_1)} R_{ijkl} \otimes \dots \otimes \nabla_{b_1 \dots b_{m_s}}^{(m_s)} R_{i'j'k'l'} \otimes \nabla_{h_1 \dots h_{p_1}}^{(p_1)} f \otimes \dots \otimes \nabla_{z_1 \dots z_{p_q}}^{(p_q)} f) \quad (1)$$

(the contractions of indices are being taken with respect to the metric  $g$ ) so that for every Riemannian manifold  $(M, g)$  and every  $f \in C^\infty(M)$ :

$$L_g(f) = \sum_{x=1}^V a_x C_g^x(f) \quad (2)$$

In order to define *conformally invariant* differential operators, we also need a notion of weight for the operators in the form (2): For any complete contraction in the form (1), we define its *weight*  $K$  to be the number  $K = -\sum_{k=1}^s (m_k + 2) - \sum_{k=1}^q p_k$ . Observe that complete contractions  $C_g(f)$  with weight  $K$  have the property that for every  $t \in \mathbb{R}$  they must satisfy  $C_{t^2g}(f) = t^K C_g(f)$ .

**Definition 2** An *intrinsic operator*  $L_g(f)$  that can be written as a linear combination  $L_g(f) = \sum_{x=1}^V a_x C_g^x(f)$  of complete contractions in the form (1) with weight  $-K$  will be called an *intrinsic operator of weight  $-K$* .

Now, recall that two metrics  $g, g'$  defined over a manifold  $M$  are *conformally equivalent* if there exists a function  $\phi \in C^\infty(M)$  so that  $g = e^{2\phi}g'$ . Now given any metric  $g$  defined over a manifold  $M$ , the set of metrics  $g'$  that are conformally equivalent to  $g$  define an equivalence class, which we denote by  $[g]$ .

A natural question is then to determine which Riemannian differential operators  $L_g(f)$  exhibit invariance properties under conformal transformations of the metric  $g$ . In the setting of differential operators, the natural generalization of the function space  $C^\infty(M)$  is the bundle  $E[w]$  of conformal densities of a given weight  $w \in \mathbb{R}$ .

**Definition 3** *Given any conformal class  $(M, [g])$  we define a  $w$ -density  $f_w$  (of weight  $w$ ) to be a function*

$$f_w : M \times [g] \longrightarrow \mathbb{R}$$

*so that for any pair  $g_1, g_2 \in [g]$  with  $g_2 = e^{2\phi} g_1$  we have that:*

$$f_w(x, g_2) = e^{w \cdot \phi} f_w(x, g_1)$$

*We denote the bundle of densities of weight  $w$  by  $E[w]$ . (Note that sometimes  $E[w]$  also denotes the space of sections of this bundle).*

**Definition 4** *An intrinsic differential operator  $L_g(f)$  will be called conformally invariant of bi-degree  $(a, b)$  if for every  $\Omega > 0$ ,  $\Omega \in C^\infty(M^n)$  we have:*

$$L_{\Omega^2 g}(\Omega^a f) = \Omega^b L_g(f).$$

In more formal language, we can say that an intrinsic operator  $L_g$  of bi-degree  $(a, b)$  maps  $E[a]$  into  $E[b]$ , in the sense that if  $f_w$  belongs to the bundle  $E[a]$  then  $L_g(f_w)$  will be an element of the bundle  $E[b]$ .

Examples of conformally invariant operators on densities have been known for some time. The most classical example is the conformal Laplacian:

$$\Delta_g^c : E[-\frac{n}{2} + 1] \rightarrow E[-\frac{n}{2} - 1].$$

which in dimension  $n$  is given by the formula:

$$\Delta_g^c = [\Delta_g + \frac{n-2}{2(n-1)} S_g]$$

where  $\Delta_g$  is the Laplace-Beltrami operator and  $S_g$  is the scalar curvature.

In 1984 Paneitz showed that for  $n = 4$  one can add lower order terms to  $\Delta_g^2$  and make it conformally invariant of bi-degree  $(0, -4)$ . Branson [7] later generalized this result to arbitrary dimensions  $n \geq 4$ . He showed that the following operator is conformally invariant of bi-degree  $(-\frac{n}{2} + 2, -\frac{n}{2} - 2)$ .

$$P_4^n \psi = \Delta_g^2 \psi - \operatorname{div}(a_n S_g g d\psi + b_n Ric_g d\psi) + \frac{n-4}{2} Q_g^n \psi \quad (3)$$

where  $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$  and  $b_n = -\frac{4}{n-2}$  and also  $Q_g^n = -\frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2$ . Here  $Ric_g$  is the Ricci curvature.

Finally, the authors in [14] proved that in odd dimensions, one can add lower order terms to the  $k^{th}$  power of the Laplacian,  $\Delta^k$ , to obtain a conformally invariant operator  $P_{2k}^n$  of bi-degree  $(-\frac{n}{2} + k, -\frac{n}{2} - k)$ .  $P_2^n$  is then  $\Delta^c$  and  $P_4^n$  is the Paneitz operator. For even dimensions, the same construction goes through provided  $k \leq \frac{n}{2}$  (see also [17] for a non-existence theorem which shows that this result is sharp). These operators are now called the GJMS operators.

It is known (see [19]) that in conformally flat space, the powers of the Laplacian are the only nontrivial linear conformally invariant operators. In section 4 we provide a general construction of conformally invariant operators in general curved spaces, and in sections 6 and 7 we show that under certain restrictions, all invariant operators arise via our construction.

### 3 Notational Conventions

$\mathbb{Z}_+$  will stand for the set of *strictly positive* integers. Throughout this paper  $n$  will stand for the (fixed) dimension in which we are considering our operators, and  $g$  will stand for a metric tensor of an  $n$ -dimensional manifold  $M$ . When we wish to consider operators for dimensions  $N \neq n$ , we will be explicitly writing out  $g^N$ . *Note:* We will be assuming that  $n \geq 3$  throughout the paper. The case  $n = 3$  is slightly different from the other cases. We will be adding special footnotes regarding the case  $n = 3$  whenever necessary.

Throughout this paper, we will be writing out linear combinations of complete contractions; it will be useful to impose certain conventions regarding the form of the factors in these complete contractions.

First, when we write  $\nabla^{(m)}$ , then  $m$  will stand for the number of differentiations. If we write  $\nabla^a$ , then  $a$  will be a raised index. Furthermore, we will usually be writing  $\nabla^{(m)}R$  for the  $m^{th}$  covariant derivative of the curvature tensor without writing out the indices of this tensor (i.e. we will not be writing out  $\nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$ ). However, we impose the restriction that when a factor  $\nabla^{(m)} R_{ijkl}$  appears in a complete contraction, then the indices  $i, j, k, l$  are *not* contracting against each other.

Furthermore, for any linear combination in the form  $\sum_{h \in H} a_h C_g^h(f)$  and any subset  $H' \subset H$ , we will call  $\sum_{h \in H'} a_h C_g^h(f)$  a *sublinear combination* of  $\sum_{h \in H} a_h C_g^h(f)$ .

Now, an important note regarding the notion of identities “holding”: The operators  $L_g(f)$  that we will be considering will be functions of a metric  $g$  and a function  $f$ . They will be written out as linear combinations of complete contractions. Now, when we write  $\sum_{h \in H} a_h C_g^h(f) = \sum_{p \in P} a_p C_g^p(f)$ , we will mean that for any manifold  $M$ , any metric  $g$ , any  $f \in C^\infty(M)$  and any  $x_0 \in M$ , the two sides of this equation have the same values at  $x_0$ . Thus, throughout this paper, when we prove that a sublinear combination  $\sum_{h \in H'} a_h C_g^h(f)$  in  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$  is equal to some expression  $\sum_{t \in T} a_t C_g^t(f)$ , we will be free to *replace* the sublinear combination  $\sum_{h \in H'} a_h C_g^h(f)$  in  $L_g(f)$  by the linear

combination  $\sum_{t \in T} a_t C_g^t(f)$ .

## 4 The construction of the new operators.

The only piece of background needed for the construction of the operators is the *ambient metric*, introduced by Fefferman and Graham in [11]. The ambient metric is a formal construction that invariantly associates to each conformal class  $(M, [g])$  an  $(n+2)$ -dimensional pseudo-Riemannian manifold  $(\tilde{G}, \tilde{g})$  (more precisely, a jet of an  $(n+2)$ -metric  $\tilde{g}$ ). It was this tool that was used, albeit in a different manner, in [14] to construct the conformally invariant powers of the Laplacian.

### 4.1 The Fefferman-Graham ambient metric.

All the material presented in this subsection comes from [11] and [12]. Let  $(M, [g])$  be a conformal class and  $g$  a representative of this class.

Define  $G = \mathbb{R}_+ \times M$ . Any coordinate patch  $U \subset M$  (with coordinates  $x^1, \dots, x^n$ ) defines a coordinate patch  $\mathbb{R}_+ \times U$  in  $G$  (with coordinates  $t, x^1, \dots, x^n$ ). Define a symmetric  $(0, 2)$ -tensor  $g^{n+1}$  on  $G$  via the formula:

$$\sum_{i,j=0}^n g_{ij}^{n+1}(t, x) dx^i dx^j = t^2 \sum_{i,j=1}^n g_{ij}(x) dx^i dx^j$$

(hence, the  $t$ -direction is null). Now let  $\tilde{G} = G \times (-1, 1)$ , where  $\tilde{G}$  has coordinates  $\{t, x^1, \dots, x^n, \rho\}$  and also  $G = \{\rho = 0\}$ . Fefferman and Graham have proven that *if the dimension  $n$  is odd* then there exists a metric  $\tilde{g}_{ij}$  (any value  $k, 1 \leq k \leq n$  of the indices  $i, j$  corresponds to the vector  $\frac{\partial}{\partial x^k}$  and the values 0 and  $n+1$  correspond to the vectors  $\frac{\partial}{\partial t}, \frac{\partial}{\partial \rho}$ ) on  $\tilde{G}$  with the following properties:

1.  $\tilde{g}(t, x, 0)|_{TG} = g^{n+1}(t, x)|_{TG}$ .
2. For  $1 \leq i, j \leq n$ ,  $\tilde{g}_{ij}(t, x, \rho) = t^2 \tilde{g}_{ij}(1, x, \rho)$ .
3. Off of the hypersurface  $\{\rho = 0\}$ , we have  $Ric(\tilde{g})(t, x, \rho) = 0 + O(\rho^\infty)$ .

Furthermore, there exists a coordinate system  $\{t, x, \rho\}$  on  $\tilde{G}$  for which the metric  $\tilde{g}_{ij}(t, x, \rho)$  can be written in a special form: Denoting  $dx^0 = dt$  and  $dx^{n+1} = d\rho$ , we have:

$$\sum_{i,j=0}^{n+1} \tilde{g}_{ij} dx^i dx^j = t^2 \sum_{i,j=1}^n g_{ij}(x, \rho) dx^i dx^j + 2t dt d\rho + 2\rho dt^2 \quad (4)$$

For each ambient metric construction, where we start off with  $(M, g)$  and perform the above construction, we will call this coordinate system  $(t, x, \rho)$  the *special coordinate system* that corresponds to  $(M, g)$ .

Fefferman and Graham have then shown in [11] that *for  $n$  odd*, the Taylor expansion of the metric  $\tilde{g}$  off of the hypersurface  $\{\rho = 0\}$  *is uniquely determined by the above requirements*. Thus, for a given metric  $g \in [g]$ , two different ambient metric constructions differ by terms in  $O(\rho^\infty)$ . Moreover, the ambient metric is a *conformally invariant construction*, in the following sense: Let  $g_1, g_2 \in [g]$ , where  $g_1 = e^{2\phi} g_2$ . Let  $(\tilde{G}_1, \tilde{g}_1), (\tilde{G}_2, \tilde{g}_2)$  be ambient metric constructions for  $g_1, g_2$ . We denote by  $(t, x^1, \dots, x^n, \rho)$  the special coordinate system that corresponds to  $\tilde{g}_1$  and by  $(t', x^1, \dots, x^n, \rho')$  the special coordinate system that corresponds to  $\tilde{g}_2$ . Then there exists a map  $\Phi : \tilde{G}_2 \rightarrow \tilde{G}_1$  so that:

1.  $\Phi(1, x^1, \dots, x^n, 0) = (e^{\phi(x^1, \dots, x^n)}, x^1, \dots, x^n, 0)$  and  $\Phi$  maps the set  $\{\rho' = 0\}$  onto the set  $\{\rho = 0\}$ .
2.  $\Phi$  respects the homogeneity of  $\tilde{G}_1$  and  $\tilde{G}_2$ , in the sense that if  $\Phi(t', x, \rho') = (t, x, \rho)$  then  $\Phi(\lambda \cdot t', x, \rho') = (\lambda \cdot t, x, \rho)$ .
3.  $(\Phi^* \tilde{g}_1) = \tilde{g}_2 + O(\rho^\infty)$  (so the ambient metric constructions for  $g_1, g_2$  are isometric mod  $O(\rho^\infty)$ ).

When  $n$  is even the ambient metric construction can only be carried out to finite order. With the notational conventions introduced above, Fefferman and Graham have shown that there exists a metric  $\tilde{g}$  on  $\tilde{G}$  so that:

1.  $\tilde{g}(t, x, 0)|_{TG} = g^{n+1}(t, x)|_{TG}$ .
2. For  $1 \leq i, j \leq n$ ,  $\tilde{g}_{ij}(t, x, \rho) = t^2 \tilde{g}_{ij}(1, x, \rho)$ .
3. Off of the hypersurface  $G^{n+1} = \{\rho = 0\}$ , we have that:  $Ric(\tilde{g})(t, x, \rho) = 0 + O(\rho^{\frac{n-4}{2}})$ , while components of  $Ric(\tilde{g})(t, x, \rho)$  that are tangential to  $G^{n+1}$  vanish to order  $\frac{n-2}{2}$ .

Furthermore, there exists a coordinate system  $\{t, x, \rho\}$  on  $\tilde{G}$  for which the metric  $\tilde{g}_{ij}(t, x, \rho)$  can be written in a special form: Denoting  $dx^0 = dt$  and  $dx^{n+1} = d\rho$ , we have:

$$\sum_{i,j=0}^{n+1} \tilde{g}_{ij} dx^i dx^j = t^2 \sum_{i,j=1}^n g_{ij}(x, \rho) dx^i dx^j + 2t dt d\rho + 2\rho dt^2 + O(\rho^{\frac{n}{2}}) \quad (5)$$

Fefferman and Graham have then shown in [11] that *for  $n$  even*, the Taylor expansion of the metric  $\tilde{g}^{n+2}$  off of the hypersurface  $\rho = 0$  *is uniquely determined by the above requirements up to order  $\frac{n}{2}$* . Thus, for a given metric  $g \in [g]$ , two different ambient metric constructions differ by terms in  $O(\rho^{\frac{n}{2}})$ .

We also note that if we denote by  $\tilde{R}_{ijkl}$  the ambient curvature tensor and by  $\tilde{\nabla}$  the Levi-Civita connection of the metric  $\tilde{g}$ , then  $\tilde{R}_{ijkl}$  is related to the curvature tensor  $R_{ijkl}$  of the underlying manifold  $(M, g)$  in a simple way:

Recall the Schouten and Weyl tensors:



$$P_{ij} = \frac{1}{n-2} [Ric_{ij} - \frac{S}{2(n-1)} g_{ij}] \quad (6)$$

$$W_{ijkl} = R_{ijkl} - [P_{jk}g_{il} + P_{il}g_{jk} - P_{ik}g_{jl} - P_{jl}g_{ik}] \quad (7)$$

Fefferman and Graham, [11], have shown that at each point  $(t, x)$  of  $G^{n+1}$ , we have that for the vectors  $X^0, X^1 \dots X^n, X^\infty$  that correspond to the coordinates  $t, x^1, \dots, x^n, \rho$ :

1.  $\tilde{R}_{ijk0}(t, x, 0) = 0, 0 \leq i, j, k \leq n+1.$
2.  $\tilde{R}_{ijkl}(t, x, 0) = t^2 W_{ijkl}(x), 1 \leq i, j, k, l \leq n.$
3.  $\tilde{R}_{ijk\infty}(t, x, 0) = t^2 C_{kij}(x), 1 \leq i, j, k.$
4.  $\tilde{R}_{\infty ij\infty}(t, x, 0) = \frac{t^2}{n-4} B_{ij}(x), 1 \leq i, j \leq n, n \neq 4$

where  $C_{kij}$  is the Cotton tensor,  $C_{kij} = \nabla_i P_{jk} - \nabla_j P_{ik}$  and  $B_{ij}$  is the Bach tensor,  $B_{ij} = C_{ijk}{}^k - P^{kl} W_{kijl}$ .

Moreover, it has been shown in [15] and that the Christoffel symbols  $\tilde{\Gamma}_{ij}^k(\tilde{x}_0)$  (with respect to the special coordinate system) are related to the underlying geometry of  $(M, g)$  by simple relations: Let the indices  $a, b, c$  take values between 1 and  $n$ . Then at each point  $\tilde{x}_0 = (1, x_0, 0) \in \tilde{G}$ ,  $x_0 \in M$ , we have:

$$\tilde{\Gamma}_{bc}^a(\tilde{x}_0) = \Gamma_{bc}^a(x_0), \tilde{\Gamma}_{ab}^\infty(\tilde{x}_0) = g_{ab}(x_0), \tilde{\Gamma}_{ab}^0(\tilde{x}_0) = -P_{ab}(x_0) \quad (8)$$

The rest of the Christoffel symbols can be computed using the formula  $\frac{\partial \tilde{g}_{ij}}{\partial \rho}(1, x, 0) = 2P_{ij}(x)$ , see [15], and the form (4) of the ambient metric at  $(t, x, 0)$ .

## 4.2 The construction of the New Operators.

In this subsection we will construct the new operators, and explain some of their features. It should be noted that the method we use is the standard way one constructs conformally invariant scalar objects using the ambient metric (initiated in [11]). We will initially construct the new operators in the odd-dimensional case when  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ . We will then explain how the method can be carried over to even dimensions and/or  $(w + \frac{n}{2}) \in \mathbb{Z}_+$ .

We start with any conformal equivalence class  $(M, [g])$  and a density  $f_w(x, g)$  of weight  $w$  defined over  $[g]$ . We pick any fixed metric  $g$  from the class  $[g]$ . Evaluating the density  $f_w$  at the metric  $g$  defines a scalar function  $f(x) = f_w(x, g)$ ,  $f \in C^\infty(M)$ .

We perform the ambient metric construction  $(\tilde{G}, \tilde{g})$  for  $(M, g)$ . In the ambient metric setting, the density  $f_w$  can be naturally viewed as a homogeneous function  $u_w(t, x)$  defined on  $G \subset \tilde{G}$  by setting:

$$u_w(t, x) = t^w f(x) \quad (9)$$

We then seek to invariantly extend this homogeneous function  $u_w(t, x)$  to a function  $\tilde{u}_w(t, x, \rho)$  defined on  $(\tilde{G}, \tilde{g})$ . We do this by requiring the extension  $\tilde{u}_w$  to be homogeneous and harmonic to infinite order off of  $\{\rho = 0\}$ :

1. For the special coordinate system  $(t, x, \rho)$ , we require that  $\tilde{u}_w(t, x, \rho) = t^w \tilde{u}(1, x, \rho)$ .
2. We require that  $\tilde{u}_w(t, x, 0) = u_w(t, x)$ .
3. We require that:

$$\Delta_{\tilde{g}} \tilde{u}_w(t, x, \rho) = O(\rho^\infty) \quad (10)$$

Here  $\Delta_{\tilde{g}}$  stands for the Laplace-Beltrami operator with respect to the ambient metric  $\tilde{g}$  (actually if  $g$  has Riemannian signature  $\Delta_{\tilde{g}}$  is the wave operator).

It is then known from [14] that if  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$  then the above equation has a unique solution  $\tilde{u}_w(t, x, \rho)$ , up to functions that vanish to infinite order off of  $G$ . Hence, we have that the covariant derivatives  $\tilde{\nabla}^{(p)} \tilde{u}_w(t, x, 0)$  at any point  $(t, x, 0) \in \tilde{G}$  are all well-defined. We will refer to the function  $\tilde{u}_w$  as the *harmonic extension of  $u_w$  to  $\tilde{G}$* .

Now, choose natural numbers  $r, K \in \mathbb{N}$ , and consider any complete contraction:

$$\text{contr}(\tilde{\nabla}^{(k_1)} \tilde{R} \otimes \dots \otimes \tilde{\nabla}^{(k_s)} \tilde{R} \otimes \tilde{\nabla}^{(l_1)} \tilde{u}_w \otimes \dots \otimes \tilde{\nabla}^{(l_r)} \tilde{u}_w), \quad (11)$$

subject to the only restriction that  $\sum l_i + \sum (k_i + 2) = K$ .

Then, for any finite set of such complete contractions,  $\{\tilde{C}_{\tilde{g}}^1(\tilde{u}_w), \dots, \tilde{C}_{\tilde{g}}^z(\tilde{u}_w)\}$ , we can form linear combinations in the form:

$$F_g(f) = \sum_{s=1}^z a_s \tilde{C}_{\tilde{g}}^s(\tilde{u}_w) \quad (12)$$

Observe that the right hand side of the above is indeed a function of  $g$  and  $f$ , since for any point  $x \in M$  the jets of  $\tilde{g}$  and  $\tilde{u}_w$  at  $(1, x, 0) \in \tilde{G}^{n+2}$  are uniquely determined by the jets of  $g$  and  $f$  at  $x \in M$ . Moreover,

**Proposition 1** *Let  $n$  be odd and  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ . Then for any linear combination  $F_g(f)$  as above,  $F_g(f)$  is an intrinsic operator of weight  $-K$ , which is conformally invariant of bi-degree  $(w, r \cdot w - K)$ .*

*Proof:* We check that  $F_g(f)$  is an intrinsic differential operator on  $(M, g)$  by virtue of the form of the ambient curvature tensor and the Christoffel symbols of the ambient metric (in the special coordinate system that corresponds to  $g$ ).

The conformal invariance of the operator  $F_g(f)$  follows from the conformal invariance of the ambient metric construction, which we discussed above: We have to show that if we pick a metric  $e^{2\phi} g \in [g]$ , (for which the corresponding

value of the density  $f_w$  will be  $\bar{f}(x) = f_w(x, e^{2\phi}g) = e^{w\phi}f$ , and we perform the same ambient metric construction as above, then the operator  $F_{e^{2\phi}g}(\bar{f})$  will satisfy:

$$F_{e^{2\phi}g}(\bar{f}) = e^{(r \cdot w - K) \cdot \phi} F_g(f).$$

In order to see this, we denote by  $(\tilde{G}_2, \tilde{g}_2)$  the ambient metric construction that corresponds to the metric  $e^{2\phi}g$ , and by  $\bar{u}_w$  the homogeneous and harmonic extension of  $\bar{f}$  to  $\tilde{G}_2$  (as in (9)). The discussion from the previous subsection shows us that there exists an isometry  $\Phi : \tilde{G}_2 \rightarrow \tilde{G}_1$  with the properties listed in the previous subsection. Therefore, if we denote by  $(\tilde{\nabla}^{(m)}\tilde{R})_2(1, x, 0)$  the iterated covariant derivative of the curvature tensor of  $\tilde{g}_2$  at  $(1, x, 0) \in \tilde{G}_2$ , and by  $(\tilde{\nabla}^{(m)}\tilde{R})_1(e^{\phi(x)}, x, 0)$  the iterated covariant derivative of the curvature tensor of  $\tilde{g}_1$  at  $(e^{\phi(x)}, x, 0) \in \tilde{G}_1$ , we will have that:

$$(\tilde{\nabla}^{(m)}\tilde{R})_2(1, x, 0) = (\Phi_*(\tilde{\nabla}^{(m)}\tilde{R})_2)(e^{\phi(x)}, x, 0) = (\tilde{\nabla}^{(m)}\tilde{R})_1(e^{\phi(x)}, x, 0) = e^{2\phi(x)}(\tilde{\nabla}^{(m)}\tilde{R})_1(1, x, 0)$$

We similarly observe that  $(\Phi_*\bar{u}_w)$  must still be a homogeneous harmonic function in  $\tilde{G}_1$ . Therefore we derive  $\Phi_*\bar{u}_w = \tilde{u}_w + O(\rho^\infty)$ .

Therefore, we have that:

$$F_{e^{2\phi}g}(\bar{f})(x) = \sum_{s=1}^z a_s \tilde{C}_{\tilde{g}_1}^s(\tilde{u}_w)(e^{w\phi(x)}, x, 0)$$

No, using the fact that  $\tilde{g}_1^{ij}(t, x, 0) = t^{-2}\tilde{g}_1^{ij}(1, x, 0)$ ,  $(\tilde{\nabla}^{(m)}\tilde{R})_1(t, x, 0) = t^2(\tilde{\nabla}^{(m)}\tilde{R})_1(1, x, 0)$ ,  $(\tilde{\nabla}^{(p)}\tilde{u}_w)_1(t, x, 0) = t^w(\tilde{\nabla}^{(p)}\tilde{u}_w)_1(1, x, 0)$ , we derive our proposition.  $\square$

**Definition 5** *A differential operator in the form (12) will be called a Weyl operator of weight  $-K$  and  $f$ -homogeneity  $r$  in  $\tilde{u}_w$ .*

*The case of half-integer weights and/or even dimensions:* If  $n$  is odd, then for each weight  $w = -\frac{n}{2} + k, k \in \mathbb{Z}_+$  it is shown in [14] that the equation (10) has a uniquely defined solution up to the order  $k - 1$ . Thus the same proof as above shows that the operators in (12) are well-defined and conformally invariant, provided that for each  $\tilde{C}_{\tilde{g}}^s(\tilde{u}_w)$  (in the form (24)) we have that  $l_h \leq k - 1$ .

If  $n$  is even, then we have seen that the ambient metric is well-defined only up to order  $\frac{n}{2}$ . Hence, the same proof as above shows that for each weight  $w, (w + \frac{n}{2}) \notin \mathbb{Z}_+$ , the operators in (12) are well-defined provided that, when the expressions (11) are written out in terms of coordinate derivatives of the ambient metric, no derivatives  $\frac{\partial^{a_1} \dots \partial^{a_n} \partial^J \tilde{g}^{n+2}}{\partial x_1^{a_1} \dots \partial x_n^{a_n} \partial \rho^J}$  with  $J \geq \frac{n}{2}$  appear. In the case where  $n$  is even and the weight  $w = -\frac{n}{2} + k, k \in \mathbb{Z}_+$ , it follows that the operators in (12) are well-defined provided the above restriction holds and also provided that for each  $\tilde{C}_{\tilde{g}}^s(\tilde{u}_w)$  we have  $l_h \leq k - 1$ .

In section 5 we will claim that all conformally invariant differential operators arise via the above construction, subject to some restrictions that we will explain. For now, we will discuss certain interesting features of the new operators:

### 4.3 Features and Examples.

Let us first illustrate how each expression (when it is well-defined):

$$F_g(f) = \sum_{s=1}^z a_s \tilde{C}_{\tilde{g}}^s(\tilde{u}_w)$$

can be seen as a 1-parameter family of operators, parametrized by the weight  $w$ . We pick a fixed metric  $g \in [g]$ , and for each  $w, (w + \frac{n}{2}) \notin \mathbb{Z}_+$  we construct the operator  $F_g(f)$  above (it will be conformally invariant of bi-degree  $(w, r \cdot w - K)$ ). We claim that  $F_g(f)$  can be expressed in the form:

$$\sum_{s=1}^z a_s \tilde{C}_{\tilde{g}}^s(\tilde{u}_w) = \sum_{h \in H} b_h(w, n) C_g^h(f) \quad (13)$$

where each  $C_g^h(f)$  is Riemannian operator of weight  $-K$  and  $f$ -homogeneity  $r$ , while  $b_h(w, n)$  is a rational function in the weight  $w$  and the dimension  $n$ .

*Proof:* This follows by calculating the Taylor expansion of  $\tilde{u}_w$  off of  $\{\rho = 0\}$ . As discussed in [14], it follows that the different components  $\tilde{\nabla}^{(m)} \tilde{u}_w$  have coefficients that are rational functions in  $n, w$ .

We furthermore notice that each complete contraction in the form (11) with  $s > 0$  will vanish if  $g$  is locally conformally flat, since in that case  $\tilde{R}_{ijkl} = 0$  (see [11]). The only complete contractions of the form (11) that do not vanish in conformally flat space are the ones for which  $s = 0, p > 1$ . The operators that arise thus are nonlinear and have already appeared in [4].

Moreover, since  $\tilde{u}_w$  is defined to be harmonic and  $\tilde{R}$  Ricci-flat, not all the contractions (11) are non-zero. On the other hand, (for  $n$  odd and  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ , for simplicity) we can easily construct a *non-zero* operator with a leading order symbol  $C^h(\tilde{g}) \Delta_g^r f$  where  $C^h(\tilde{g})$  is one of the conformally invariant scalars originally constructed in [11]:

Consider any linear combination  $P(g) = \sum_{s=1}^z a_s \tilde{C}^s(\tilde{g})$  where each  $\tilde{C}^s(\tilde{g})$  is in the form:

$$\text{contr}(\tilde{\nabla}^{(m_1)} \tilde{R} \otimes \dots \tilde{\nabla}^{(m_s)} \tilde{R})$$

with a given weight  $-K$ . These Riemannian scalars are the original conformal invariants constructed by Fefferman and Graham in [11]. Now, for each contraction  $\tilde{C}^s(\tilde{g})$ , we let  $\tilde{C}_{\tilde{g}}^s(\tilde{u}_w)$  stand for the complete contraction:

$$\text{contr}((\tilde{\nabla})^{t_1 \dots t_r} [\tilde{C}^s(\tilde{g})] \otimes (\nabla)_{t_1 \dots t_r}^{(r)} \tilde{u}_w)$$

By [14] it follows that  $\tilde{\nabla}_{\infty \dots \infty}^{(r)} \tilde{u}_w = C_{w,n} \Delta_g^r f + (\text{lot})$ , here  $C_{w,n}$  is a non-zero constant. Also, we have that each other component of  $\tilde{\nabla}^{(r)} \tilde{u}_w$  has order less than  $2r$ . Lastly, using the fact that  $\tilde{\Gamma}_{00}^K = 0$  (and this follows from (4)), we see that  $\tilde{\nabla}_{00 \dots 0}^{(r)} [\tilde{C}_{\tilde{g}}^s] = (-K)^r \cdot [\tilde{C}_{\tilde{g}}^s]$ . Thus we derive our claim.

To illustrate, we write out two examples of new conformally invariant differential operators arising from the formula (11). For both these examples we refer the reader to [12] for explicit computations.

Firstly, consider the ambient complete contraction:

$$L_g^1(f) = \text{contr}(\tilde{\nabla}^m |\tilde{R}_{ijkl}|^2 \otimes \tilde{\nabla}_m \tilde{u}_0)$$

Fefferman and Hirachi calculate that  $L_g^1(f)$  is of the form:

$$L_g^1(f) = \nabla^i (|W|^2) \nabla_i f + \frac{4}{n-2} |W|^2 \Delta f$$

Our second example will illustrate that not all our new operators have a leading order term  $C^h(\tilde{g}) \Delta_g^r f$ . We consider the operator  $L_g^\sharp(f)$  that arises by the complete contraction:

$$\text{contr}(\tilde{\nabla}_{st}^2 \tilde{u}_0 \otimes \tilde{R}_{jkl}^s \otimes \tilde{R}^{tjkl})$$

Fefferman and Hirachi show that  $L_g^\sharp(f)$  can be written out explicitly in the form:

$$L_g^\sharp(f) = W_{ijk}{}^l W^{ijkm} \nabla_{lm}^{(2)} f - 2C_{kij} W^{ijkl} \nabla_l f + \frac{1}{n-2} |W|^2 \Delta f \quad (14)$$

and hence its leading order term is

$$W_{ijk}{}^l W^{ijkm} \nabla_{lm}^{(2)} f + \frac{1}{n-2} |W|^2 \Delta f.$$

## 5 The completeness Results.

In this section we will present our claims that, under certain restrictions, all conformally invariant operators can be written as Weyl operators. The main restriction we need is that the weight (or the degree for theorem 3) be bounded by the dimension. In the case of even dimensions and/or half-integer weight, we will also impose additional restrictions. Since these additional restrictions are quite technical (they depend on the parameters  $\iota, \tau, \beta, \gamma$  introduced below), the reader may wish to skip the discussion of the extra restrictions at first.

All three completeness theorems 1, 2 and 3 are proven by a novel approach that we introduce in sections 6, 7. Theorem 4 below deals with the case where the weight is greater than the dimension; it will be proven in section 8 by a straightforward adaptation of the methods in [6].

In order to make our task easier, we will make an observation: Let us suppose that  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$  is conformally invariant of bi-degree  $(a, b)$ . Let us break the index set  $H$  into subsets  $H_z$  according to the rule:  $h \in H_z$  if and only if  $C_g^h(f)$  has  $q = z$  (i.e. is homogeneous of degree  $z$  in the function  $f$ ). Accordingly, we define:

$$L_g^z(f) = \sum_{h \in H_z} a_h C_g^h(f) \quad (15)$$

Just by applying the definition of conformal invariance we see that:

**Lemma 1** *In the above notation, if  $L_g(f)$  is conformally invariant of bi-degree  $(a, b)$ , then each  $L_g^z(f)$  is conformally invariant of bi-degree  $(a, b)$ .*

The above Lemma allows us to restrict our attention to linear combinations:

$$L_g(f) = \sum_{h \in H} a_h C_g^h(f)$$

where each  $C_g^h(f)$  is in the form (1) and has a fixed homogeneity  $\kappa$  in  $f$ . Thus, from now on we will assume that each  $C_g^h(f)$  has  $\kappa$  factors in the form  $\nabla^{(p)} f$ .

**Definition 6** *For each complete contraction  $C_g(f)$  in the form (1) we define  $\kappa^\sharp$  to stand for the number of factors  $\nabla^{(p)} f$  with  $p \geq 1$ . Recall  $s$  stands for the number of factors  $\nabla^{(m)} R$ . We then define  $2s + \kappa^\sharp$  to be the degree of  $C_g(f)$ ,  $\deg[C_g(f)]$ .*

It is important to observe that any complete contraction in the form (1) with weight  $-K$  will satisfy  $\deg[C_g(f)] \leq K$ . Notice also that this definition is slightly different from the one in [6].

For any complete contraction we will be paying special attention to pairs of indices that belong to the same factor and are contracting against each other. We call such pairs of indices *internal contractions* (they are called internal traces in [6]). Also, for any complete contraction  $C_g(f)$  in the form (1) and any factor  $F$  in  $C_g(f)$ ,  $\tau[F]$  will stand for the total number of internal contractions in  $F$  and  $\iota[F]$  will stand for the total number of indices in  $F$  (also counting the pairs of indices that are involved in internal contractions).

**Definition 7** *Consider any complete contraction  $C_g(f)$  in the form (1) and we pick out its  $\kappa$  factors  $F_1, \dots, F_\kappa$  in the form  $\nabla^{(p)} f$ . For each  $F_s, 1 \leq s \leq \kappa$  we define  $\beta[F_s] = \iota[F_s] - \tau[F_s]$ . We then define  $\beta[C_g(f)]$  to stand for the maximum among the numbers  $\beta[F_1], \dots, \beta[F_\kappa]$ . Finally, for a linear combination  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$  we define  $\beta[L_g(f)]$  to be  $\max_{h \in H} \beta[C_g^h(f)]$ .*

Our Theorem for odd dimensions is then the following:

**Theorem 1** *Let the dimension  $n$  be odd. We pick any numbers  $K \in 2\mathbb{Z}_+$  with  $K \leq n$ ,  $\kappa \in \mathbb{Z}_+$  and any weight  $w$ . If  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$  then any Riemannian differential operator  $L_g(f)$  of weight  $-K$ ,  $f$ -homogeneity  $\kappa$  which is conformally invariant of bi-degree  $(w, w \cdot \kappa - K)$  can be written as a Weyl operator.*

*If  $w = -\frac{n}{2} + k$  for some  $k \in \mathbb{Z}_+$  then our conclusion above holds under the extra assumption that  $\beta[L_g(f)] < k$ .*

*Even dimensions:* In order to state our theorem in this case, we introduce one more notational convention.

**Definition 8** *Given any complete contraction  $C_g(f)$  in the form (1), we list all its factors  $F_1, \dots, F_d$ . For a factor  $F_s$  in the form  $\nabla^{(m)} R_{ijkl}$ , we define  $\gamma[F_s] = \iota[F_s] - \tau[F_s] - 2$ . For a factor  $F_s$  in the form  $\nabla^{(p)} f$ , we define  $\gamma[F_s] = \beta[F_s]$ .*

*Then, for each complete contraction  $C_g^h(f)$  we define  $\gamma[C_g^h(f)]$  to stand for the maximum among the numbers  $\gamma[F_1], \dots, \gamma[F_d]$ . If  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$ , we define  $\gamma[L_g(f)] = \max_{h \in H} \gamma[C_g^h(f)]$ .*

**Theorem 2** *Let the dimension  $n$  be even. We pick any numbers  $K \in 2\mathbb{Z}_+$  with  $K \leq n$ ,  $\kappa \in \mathbb{Z}_+$  and any weight  $w$ . If  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ , then any Riemannian differential operator  $L_g(f)$  of weight  $-K$  and  $f$ -homogeneity  $\kappa$ , with  $\gamma[L_g(f)] < \frac{n}{2}$  which is conformally invariant of bi-degree  $(w, \kappa \cdot w - K)$  can be written as a Weyl operator.*

*In the case where  $w = -\frac{n}{2} + k$ ,  $k \in \mathbb{Z}_+$ , we have that the above conclusion holds under the additional assumption that  $\beta[L_g(f)] < k$ .*

Let us observe that an easy consequence of the above is that any linear conformally invariant operator  $L_g(f)$  of bi-degree  $(-\frac{n}{2} + k, -\frac{n}{2} - k)$  with  $k \leq \frac{n}{2}$  can be written in the form:

$$L_g(f) = (Const) \cdot P_g^n(f) + \sum_{h \in H} a_h \tilde{C}_{\tilde{g}^{n+2}}^h(\tilde{u}_{-\frac{n}{2}+k})$$

where  $P_g^n(f)$  is the GJMS operator constructed in [14], with leading symbol  $\Delta^k$  and  $\sum_{h \in H} \dots$  is a linear Weyl operator of bidegree  $(-\frac{n}{2} + k, -\frac{n}{2} - k)$ : We only have to observe that all complete contractions  $C_g(f)$  in the form (1) (linear in  $f$ ) with weight  $-2k (\geq -n)$  and  $s \geq 1$  factors  $\nabla^{(m)} W$  will automatically have  $\gamma[C_g(f)], \beta[C_g(f)] < k \leq \frac{n}{2}$ .

Both our Theorems above require that  $K \leq n$ , where  $K$  is the weight of the operators and  $n$  the dimension. If we wish to overcome this restriction, however, we can prove a weaker result, which will show that conformally invariant operators in general can be written as Weyl operators, plus corrections with degree that is greater than the dimension:

Consider any Riemannian operator  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$ , where each  $C_g^h(f)$  is in the form (1) with weight  $-K$  and  $f$ -homogeneity  $\kappa$ . To simplify our claim, we will assume that each factor  $\nabla^{(p)} f$  has  $p > 0$  (although this restriction can easily be overcome). For each such  $L_g(f)$ , we define its *minimum degree* to be  $\min_{h \in H} \deg[C_g^h(f)]$  (see definition 7), which we denote by  $\mindeg[L_g(f)]$ . Notice that if the minimum number of factors among the contractions  $\{C_g^h(f)\}_{h \in H}$  is  $\sigma$ , then  $2\sigma + \kappa = \mindeg[L_g(f)]$  (i.e. the minimum degree is essentially determined by the minimum number of factors). We can then show the following:

**Theorem 3** Consider a Riemannian differential operator  $L_g(f)$  of weight  $-K$ ,  $K \in 2\mathbb{Z}_+$  and  $f$ -homogeneity  $\kappa$ . Assume  $L_g(f)$  is conformally invariant of bi-degree  $(w, \kappa \cdot w - K)$ ,  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ .

Assume that  $\mindeg[L_g(f)] \leq n$ . Our conclusion is then that if  $n$  is odd,  $L_g(f)$  can be written as a Weyl operator, modulo correction terms with more factors (and therefore higher degree): We denote by  $\tilde{u}_w$  the harmonic extension of the density  $f_w$ . Then we can write:

$$L_g(f) = \sum_{h \in H'} a_h \tilde{C}_g^h(\tilde{u}_w) + \sum_{t \in T} a_t C_g^t(f) \quad (16)$$

where each  $\tilde{C}_g^h(\tilde{u}_a)$  is a Weyl operator and each  $C_g^t$  has degree  $> \mindeg[L_g(f)]$ .

In the case where  $n$  is even and/or where  $w = -\frac{n}{2} + k$  for some  $k \in \mathbb{Z}_+$ , the above is still true provided  $\gamma[L_g(f)] < \frac{n}{2}$  and/or  $\beta[L_g(f)] < k$ .

We observe that this third theorem applies even when  $K > n$ , provided that  $\mindeg[L_g(f)] \leq n$ . Thus, the theorem above can be iteratively applied, until we reach some linear combination  $\sum_{t \in T'} a_t C_g^t(f)$  on the right hand side of (16) for which  $\mindeg[\sum_{t \in T'} a_t C_g^t(f)] > n$ .

For all three theorems above, we will refer to the restrictions on  $\beta[C_g^h(f)]$  and  $\gamma[C_g^h(f)]$  (whenever we do impose restrictions on these parameters) as the *extra restrictions*. Also, we note that in the case  $n = 3$ , theorems 1, 2 can only be applied in the case where  $L_g(f) = a_1 C_g^1(f) + a_2 C_g^2(f)$ , where  $C_g^1(f) = |\nabla f|^2$  and  $C_g^2(f) = \Delta f \cdot f$  (because of the weight restrictions). On the other hand, still in the case  $n = 3$ , theorem 3 only applies when  $\kappa \leq 3$  (because of the degree restriction). In the cases  $\kappa = 2, \kappa = 3$ ,  $L_g(f)$  can only be a linear combination without curvature terms (since  $\mindeg[L_g(f)] \leq 3$ ). In the case  $\kappa = 1$ , then theorem 3 only applies for  $\mindeg[L_g(f)] = 3$ , in which case the terms of minimum degree in  $L_g(f)$  must be in the form  $\text{contr}(\nabla^{(m)} R \otimes \nabla^{(p)} f)$ . The cases  $n = 3, \kappa \geq 2$  will follow by the general argument to follow. The subcase  $\kappa = 1$  is slightly different, and we highlight this in footnotes, whenever needed.

*Theorem 3 and [8], [6]:* A differential operator  $L(f)$  (of  $f$ -homogeneity  $\kappa$  and weight  $-K$ ), defined only on Euclidean space and assumed to be invariant under the action of the orthogonal group can be written as a linear combination of contractions in the form:

$$\text{contr}(\nabla^{(p_1)} f \otimes \dots \otimes \nabla^{(p_\kappa)} f) \quad (17)$$

The additional invariance required of  $L(f)$  in [8] and [6] is invariance under the conformal group. For comparison, we will consider a Riemannian differential operator  $L_g(f)$ , whose *principal symbol* agrees with that of  $L(f)$ . However,  $L_g(f)$  may also contain complete contractions with more than  $\kappa$  factors, and these will include curvature terms. In our setting,  $L_g(f)$  is required to be conformally invariant in the sense of definition 4.



We observe that if we apply theorem 3, then in Euclidean space the correction terms will vanish, since they will involve a factor of the curvature. Thus for  $g_{Eucl}$  being the Euclidean metric, whenever theorem 3 can be applied it will show that  $L_{g_{Eucl}}(f)$  can be written as a Weyl operator, without correction terms.

Let us observe how Theorem 3 strengthens the existing results for Euclidean space (in the *more restricted setting* of Riemannian operators assumed to be conformally invariant in the sense of definition 4), provided  $\kappa \leq n$ : The case where  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$  and  $w \notin \mathbb{Z}_+$  has been settled in [8]. When  $n$  is odd and  $w = -\frac{n}{2} + k$ , theorem 3 is the first completeness result, even in Euclidean space. In the case where  $n$  is odd and  $w \in \mathbb{Z}_+$ , [6] proves that every invariant operator is a Weyl operator, but only provided each  $p_i \geq w$  for every  $p_i$  in (17). Our Theorem 3 shows that every invariant operator is a Weyl operator, i.e. it imposes no restriction on any  $p_i$  and it works for any  $w \in \mathbb{Z}_+$ .

For  $n$  even, if  $w = -\frac{n}{2} + k, k \in \mathbb{Z}_+$  (which now includes the case  $w \in \mathbb{Z}_+$ ), [6] proves all invariant operators are Weyl provided they can be expressed as polynomials in the derivatives  $\tilde{\nabla}^{(p_i)} \tilde{u}_w$ , where  $\tilde{u}_w$  solves (10) and provided each  $p_i \geq w$ . Theorem 3 works without these a priori restrictions, and only requires  $\beta[L_g(f)] < k$ .

Finally, let us also note that our methods can also strengthen the existing completeness results in [6] for invariants that depend only on the curvature (i.e. we have a linear combination of contractions in the form (1) with  $q = 0$ ). The authors in [6] show that any such conformal invariant must be Weyl provided the degree of the contractions is  $< n$  (in the notation of definition 6—this corresponds to degree  $< \frac{n}{2}$  in the notation of [6]). Theorem 3 also shows this claim for the case of degree  $n$ .

For completeness, we state a last theorem for the case  $\mindeg[L_g(f)] > n$ , which will be proven in the last section, by an adaptation of the methods in [6]:

**Theorem 4** *Consider any Riemannian operator  $L_g(f)$  with  $f$ -homogeneity  $\kappa$ , conformally invariant of bi-degree  $(w, \kappa \cdot w - K)$ . Suppose that  $n$  is odd and  $(w + \frac{n}{2}) \notin \mathbb{Z}_+, w \notin \mathbb{Z}_+, (2\kappa \cdot w + n - 2K) \notin -2\mathbb{Z}_+, (\kappa \cdot w + n - K) \notin -\mathbb{Z}_+$ .*

*Then  $L_g(f)$  can be written as a Weyl operator.*

## 6 Proof of theorems 1, 2 and 3: First half.

*Outline of the ideas:* We prove theorems 1, 2, 3 in steps. The main argument is inductive: We consider the complete contractions with the smallest number,  $\sigma$ , of factors in  $L_g(f)$  (denote this sublinear combination by  $L_g^\sigma(f)$ ) and we show that we can subtract a Weyl operator from  $L_g(f)$  to cancel out  $L_g^\sigma(f)$ , modulo introducing contractions with more than  $\sigma$  factors. Iteratively repeating this step eventually shows our theorems, since for a given weight  $-K$ , we cannot have more than  $K$  factors in our contractions.

In order to determine the Weyl operator referred to above, and also to prove the cancellation, we re-express the conformally invariant operator  $L_g(f)$  as a linear combination of contractions involving the Weyl and symmetrized Schouten

tensors (see (24)). It turns out that this decomposition is well-suited for our purposes, since this naturally decomposes the curvature into a conformally invariant and a non-conformally invariant part.

The three steps then proceed as follows: Initially we consider the contractions in  $L_g^\sigma(f)$  and among those we pick out the ones with the maximum number  $M$  of (symmetrized) Schouten tensors. If  $M > 0$ , we prove that this linear combination must vanish modulo introducing correction terms with  $\sigma + 1$  factors. This is done in Proposition 2. By repeating this argument, we are reduced to showing our claim when all the contractions in  $L_g^\sigma(f)$  have no Schouten tensors. Then the key is to look at the number of internal contractions in the terms in  $L_g^\sigma(f)$ . We first pick out the complete contractions in  $L_g^\sigma(f)$  without internal contractions. We show that we can subtract a Weyl operator from that linear combination, modulo introducing contractions that either have at least  $\sigma + 1$  factors, *or* have  $\sigma$  factors, no Schouten tensors, but at least one internal contraction in some factor (Proposition 3). The hardest part of the proof is to then show that if all contractions in  $L_g^\sigma(f)$  have no Schouten terms, *and* have at least one internal contraction, then  $L_g^\sigma(f)$  must vanish, modulo correction terms with at least  $\sigma + 1$  factors (Proposition 4). It is at this stage in the argument that we need the degree of these contractions to be bounded by the dimension:

The proof of Proposition 4 relies on another induction in the minimum number,  $\mu$ , of internal contractions among the terms in  $L_g^\sigma(f)$ . We show that the sublinear combination in  $L_g^\sigma(f)$  of the terms with  $\mu$  internal contractions can be written as a linear combination of contractions with  $\sigma$  factors (no Schouten tensors) and at least  $\mu + 1$  internal contractions (modulo corrections with  $\sigma + 1$  factors). In order to prove this claim, we examine the transformation law of the operator under conformal rescaling, and pick out a very special linear combination in that transformation law that exactly corresponds to the terms in  $L_g^\sigma(f)$  with  $\mu$  internal contractions. Using Weyl's theory we deduce that this linear combination must vanish separately. Then, using Weyl's theory again and an operation called "Weylification", we deduce that we can make the special linear combination in  $L_g(f)$  vanish, modulo introducing terms with  $\sigma$  factors and  $\mu + 1$  internal contractions (and also terms with  $\sigma + 1$  factors). Inductively repeating this argument we prove Proposition 4. (A more detailed synopsis of this last step is provided in section 7).

## 6.1 Normalizations, and the three parts of the proof.

Starting with  $L_g(f)$  we first re-write  $L_g(f)$  as a linear combination of contractions in a new form.

Let us recall a few formulas. Firstly, the curvature identity:

$$[\nabla_i \nabla_j - \nabla_j \nabla_i] X_l = R_{ijkl} X^k \quad (18)$$

Secondly, we recall the Weyl and Schouten tensors (see (6), (7) above).

The Weyl tensor is conformally invariant and trace-free:

$$W_{ijkl}(e^{2\phi(x)}g) = e^{2\phi(x)}W_{ijkl}(g) \quad (19)$$

The Schouten tensor has the following transformation law:

$$P_{ij}(e^{2\phi}g) = P_{ij}(g) - \nabla_{ij}^{(2)}\phi + \nabla_i\phi\nabla_j\phi - \frac{1}{2}\nabla^k\phi\nabla_k\phi g_{ij} \quad (20)$$

while the Levi-Civita connection transforms:

$$(\nabla_k\eta_l)(e^{2\phi}g) = (\nabla_k\eta_l)(g) - \nabla_k\phi\eta_l - \nabla_l\phi\eta_k + \nabla^s\phi\eta_sg_{kl} \quad (21)$$

and the full curvature tensor  $R_{ijkl}$  transforms:

$$\begin{aligned} R_{ijkl}(e^{2\phi(x)}g) = e^{2\phi(x)}[R_{ijkl}(g) + \nabla_{il}^{(2)}\phi g_{jk} + \nabla_{jk}^{(2)}\phi g_{il} - \nabla_{ik}^{(2)}\phi g_{jl} - \nabla_{jl}^{(2)}\phi g_{ik} \\ + \nabla_i\phi\nabla_k\phi g_{jl} + \nabla_j\phi\nabla_l\phi g_{ik} - \nabla_i\phi\nabla_l\phi g_{jk} - \nabla_j\phi\nabla_k\phi g_{il} + |\nabla\phi|^2 g_{il}g_{jk} - |\nabla\phi|^2 g_{ik}g_{lj}] \end{aligned} \quad (22)$$

We also recall the formula (for  $n > 3$ )<sup>2</sup>:

$$\nabla_a P_{bc} - \nabla_b P_{ac} = \frac{1}{n-3}\nabla^d W_{abcd} \quad (23)$$

Now, consider our Riemannian operator  $L_g(f) = \sum_{h \in H} a_h C_g^h(f)$ , where each contraction  $C_g^h(f)$  is in the form (1). *Throughout the rest of this paper we will be assuming that  $L_g(f)$  is conformally invariant of bi-degree  $(w, \kappa \cdot w - K)$ .* We will re-write  $L_g(f)$  as a linear combination of contractions involving factors  $\nabla^{(a)}f$ , differentiated Weyl tensors and also tensors of the form  $S\nabla_{r_1 \dots r_p}^{(p)} P_{ab}$  (which stands for the fully symmetrized part of the  $(p+2)$ -tensor  $\nabla_{r_1 \dots r_p}^{(p)} P_{ab}$ ). Explicitly, our “new” complete contractions will be in the form<sup>3</sup>

$$\text{contr}(\nabla^{(m_1)}W \otimes \dots \otimes \nabla^{(m_s)}W \otimes S\nabla^{(p_1)}P \otimes \dots \otimes S\nabla^{(p_r)}P \otimes \nabla^{(a_1)}f \otimes \dots \otimes \nabla^{(a_\kappa)}f) \quad (24)$$

This can be done easily: Starting with any complete contraction  $C_g^h(f)$  in  $L_g(f)$  (in the form (1)), we only have to decompose the curvature tensor as in (7) and then repeatedly apply the equations (23) and (18), to express  $C_g^h(f)$  as a linear combination of contractions in the form (24). Thus, we re-write  $L_g(f)$ :

$$L_g(f) = \sum_{u \in U} a_u C_g^u(f) \quad (25)$$

<sup>2</sup>For  $n = 3$  the Cotton tensor ( $C_{jkl} = \nabla_k P_{lj} - \nabla_l P_{kj}$ ) is conformally invariant and can be thought of as a substitute for the Weyl tensor.

<sup>3</sup>For  $n = 3$ , we recall the discussion after theorem 3. Thus, if  $L_g(f)$  contains complete contractions with curvature factors we derive that  $L_g^\sigma(f) = \sum_{b \in B_1 \cup B_2} a_b C_g^b(f)$ , where the contractions indexed in  $B_1$  are in the form  $\text{contr}(S\nabla^{(p)}P \otimes \nabla^{(y)}f)$  and the ones indexed in  $B_2$  must be in the form  $\text{contr}(\nabla^l \nabla^{(m)} C_{jkl} \otimes \nabla^{(p)}f)$ .

where each  $C_g^u(f)$ ,  $u \in U$  is in the form (24).

In order to state our Propositions below, we need a final piece of notation:

Consider any complete contraction  $C_g(f)$  in the form (24), or even more generally in the form:

$$\text{contr}(\nabla^{(m_1)}W \otimes \dots \otimes \nabla^{(m_s)}W \otimes S\nabla^{(p_1)}P \otimes \dots \otimes S\nabla^{(p_r)}P \otimes \nabla^{(\nu_1)}R \otimes \dots \otimes \nabla^{(\nu_t)}R \otimes \nabla^{(a_1)}f \otimes \dots \otimes \nabla^{(a_\kappa)}f) \quad (26)$$

For any factor  $F_h = \nabla^{(p)}f$  or  $F_h = \nabla^\nu R$ , we define  $\beta[F_h]$ ,  $\gamma[F_h]$  as in the case of complete contractions in the form (1). For any factor  $F_h$  in the form  $\nabla^{(m)}W_{ijkl}$ , we define  $\gamma[F_h] = \iota[F_h] - \tau[F_h] - 2$ . Also, for any factor  $F_h$  of the form  $S\nabla^{(p)}P$ , we define  $\gamma[F_h] = \iota[F_h] - \tau[F_h] - 1$ . (Recall that  $\iota[F_h]$  stands for the total number of indices in  $F_h$  and  $\tau[F_h]$  stands for the total number of internal contractions in  $F_h$ ).

In general, for any complete contraction  $C_g(f)$  in the form (24) or (26), we define  $\beta[C_g(f)]$  to be the maximum among the numbers  $\beta[F_h]$ , for factors  $F_h$  in the form  $\nabla^{(p)}f$ . We also define  $\gamma[C_g(f)]$  to be the maximum of the numbers  $\gamma[F_h]$ , where  $F_h$  can be any factor in  $C_g(f)$ .

A technical tool that will be useful further down is the following:

**Observation 1** *Consider a Riemannian differential operator  $L_g(f)$ , expressed in the form:*

$$L_g(f) = \sum_{h \in H} a_h C_g^h(f) \quad (27)$$

where each  $C_g^h(f)$  is in the form (1). Suppose that for each  $h \in H$  we have  $\beta[C_g^h(f)] \leq \tau_1$ ,  $\gamma[C_g^h(f)] \leq \tau_2$ .

Then, as explained above, we write  $L_g(f)$  as a linear combination:

$$L_g(f) = \sum_{h \in H'} a_h C_g^h(f) \quad (28)$$

where each  $C_g^h(f)$  is in the form (24). It follows that for each  $h \in H'$  we have  $\beta[C_g^h(f)] \leq \tau_1$ ,  $\gamma[C_g^h(f)] \leq \tau_2$ . The converse is also true.

In view of the above, we may consider a Riemannian operator and write it as a linear combination of contractions in either of the forms (1), (24) and unambiguously say  $\beta[L_g(f)] \leq \tau_1$  and/or  $\gamma[L_g(f)] \leq \tau_2$ . Thus, we see that since  $L_g(f)$  fulfils the extra restrictions (when they are applicable) when written as a linear combination of contractions in the form (1), it still fulfils the extra restrictions when written as a linear combination of contractions in the form (24).

Now, to prove our Theorems 1, 2, 3 we consider  $L_g(f) = \sum_{u \in U} a_u C_g^u(f)$ , (each  $C_g^u(f)$  in the form (24)), and we break the index set  $U$  into subsets:

Firstly, we pick out the complete contractions  $C_g^u(f)$  (in the form (24)) with the minimum number of factors, say  $\sigma$ .

We index the contractions with  $\sigma$  factors in  $U_\sigma \subset U$ . We then further subdivide  $U_\sigma$  into subsets  $U_{\sigma,a}$ ,  $a = 0, 1, \dots, \sigma - \kappa$  according to the number of factors  $S\nabla^{(p)}P$  in  $C_g^u(f)$ : We say  $u \in U_{\sigma,a}$  if  $C_g^u(f)$  (in the form (24)) contains  $a$  factors  $S\nabla^{(p)}P$ .

Our first Proposition is then the following:

**Proposition 2** *Consider the maximum  $a$  for which  $U_{\sigma,a} \neq \emptyset$ , and denote it by  $a_M$ . Suppose  $a_M > 0$ . We then claim that we can write:*

$$\sum_{u \in U_{\sigma,a_M}} a_u C_g^u(f) = \sum_{j \in J} a_j C_g^j(f) \quad (29)$$

where each  $C_g^j(f)$  is a complete contraction in the form (1) with at least  $\sigma + 1$  factors. Furthermore each  $C_g^j(\phi)$  satisfies the extra restrictions  $\beta[C_g^j(\phi)] < k$  and  $\gamma[C_g^j(\phi)] < \frac{n}{2}$ , whenever these restrictions are applicable.

We observe that if we can prove the above we may just replace the sublinear combination  $\sum_{u \in U_{\sigma,a_M}} a_u C_g^u(f)$  in  $L_g(f)$  by the right hand side of (29). Thus, if we can prove Proposition 2, by iterative repetition we reduce ourselves to showing our theorems under the additional assumption that each  $C_g^u(f)$ ,  $u \in U_\sigma$  has no factors  $S\nabla^{(p)}P$ .

Under this assumption, we define  $U_\sigma^{0,*} \subset U_\sigma^0$  to stand for the index set of complete contractions in the form (24) with length  $\sigma$ , no factors  $S\nabla^{(p)}P$  and also no internal contractions among any of its factors. We claim:

**Proposition 3** *Suppose that the maximum  $a$  for which  $U_{\sigma,a} \neq \emptyset$  is 0. Consider the sublinear combination  $\sum_{u \in U_{\sigma,0}^*} a_u C_g^u(f)$ ; we claim that we can construct a Weyl operator  $\sum_{u \in U_{\sigma,0}^*} a_u \tilde{C}_g^u(\tilde{u}_w)$ , so that:*

$$\sum_{u \in U_\sigma^{0,*}} a_u C_g^u(f) - \sum_{u \in U_\sigma^{0,*}} a_u \tilde{C}_g^u(\tilde{u}_w) = \sum_{u \in U'} a_u C_g^u(f) + \sum_{j \in J} a_j C_g^j(f) \quad (30)$$

Here each  $C_g^u(f)$ ,  $u \in U'$ , is in the form (24) with  $\sigma$  factors, each in the form  $\nabla^{(m)}W$  or  $\nabla^{(p)}f$  and at least one of which has an internal contraction. The contractions  $C_g^j(f)$  are as in the previous Proposition. Moreover, the contractions on the RHS satisfy the extra restrictions whenever they are applicable.

Clearly, if we can show the above we will be reduced to showing our Theorems in the case where each  $C_g^u(f)$  with  $\sigma$  factors in  $L_g(f)$  has no factors  $S\nabla^{(p)}P$  and also has at least one internal contraction. Under that assumption, we then break up the index set  $U_\sigma$  into subsets  $U_\sigma^\delta$ , according to the rule that  $u \in U_\sigma^\delta$  if and only if  $C_g^u(f)$  has  $\delta$  internal contractions. We then have our last and hardest claim:

**Proposition 4** *Suppose that all the contractions with  $\sigma$  factors in  $L_g(f) = \sum_{u \in U} a_u C_g^u(f)$  are in the form (24) with no factors  $S\nabla^{(p)}P$ . Suppose also that the minimum  $\delta$  for which  $U_\sigma^\delta \neq \emptyset$  is  $\mu > 0$ . We claim that we can write:*

$$\sum_{u \in U_\sigma^\mu} a_u C_g^u(f) = \sum_{u \in U'} a_u C_g^u(f) + \sum_{j \in J} a_j C_g^j(f) \quad (31)$$

Here each  $C_g^u(f)$ ,  $u \in U'$ , is in the form (24) with  $\sigma$  factors, each in the form  $\nabla^{(m)}W$  or  $\nabla^{(p)}f$  and with at least  $\mu + 1$  internal contractions. The contractions on the RHS satisfy the extra restrictions whenever they are applicable. The linear combination indexed in  $J$  is as in Proposition 2.

Clearly, there is an obvious upper bound on the number of internal contractions for any complete contraction of weight  $-K$  (for example  $K$ ). Therefore, if we can show the above Lemma then by iterative repetition we will be reduced to the case where each complete contraction in  $L_g(f)$  has at least  $\sigma + 1$  factors.

Now, also observe that there is an obvious upper bound on the total number of factors for any complete contraction of weight  $-K$  and  $f$ -homogeneity  $\kappa$  (say  $K + \kappa$ ). Therefore, if we can prove the above three Propositions, by iterative repetition we derive our Theorems 1, 2, 3<sup>4</sup>.

## 6.2 Proof of the Propositions 2, 3.

*General discussion:* Our proof of Proposition 2 will rely on a simple study of the transformation laws of the complete contractions involved, under conformal changes of the underlying metric.

**Definition 9** *Given any pair of numbers  $(a, b)$ , any Riemannian operator  $L_g(f)$  and any  $\phi \in C^\infty(M)$  we define  $Im_\phi^{Z|(a,b)}[L_g(f)]$  as follows:*

$$Im_\phi^{Z|(a,b)}[L_g(f)] = \frac{\partial^Z}{\partial \lambda^Z} \Big|_{\lambda=0} \{ e^{-b\lambda\phi} C_{e^{2\lambda\phi}g}(e^{a\lambda\phi}f) \} \quad (32)$$

We straightforwardly observe that  $Im_\phi^{Z|(a,b)}[L_g(f)]$  is just the linear combination of summands in  $e^{-b\lambda\phi} C_{e^{2\lambda\phi}g}(e^{a\lambda\phi}f)$  that are homogenous of degree  $Z$  in the function  $\phi$ .

We make an important observation that will be used often below: Consider any operator  $L_g(f) = \sum_{u \in U} a_u C_g^u(f)$  which is conformally invariant of bi-degree  $(a, b)$ . Then, for any  $Z \geq 1$  and any function  $\phi \in C^\infty(M)$ , we must have:

$$(Im_\phi^{Z|(a,b)}[L_g(f)] =) Im_\phi^{Z|(a,b)} \left[ \sum_{u \in U} a_u C_g^u(f) \right] = 0 \quad (33)$$

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<sup>4</sup>For the case  $n = 3$  with  $\kappa = 1$ , in the notation of the footnote 3 on page 19, the argument of Proposition 4 will show that  $\sum_{b \in B_2} a_b C_g^b(f) = 0$  modulo contractions with three factors. In conjunction with Proposition 2 (which holds as stated for  $n = 3$  and will show that  $\sum_{b \in B_1} a_b C_g^b(f) = 0$  modulo complete contractions with 3 factors), that will prove theorem 3 in this case.

This essentially just follows from the definition of conformal invariance.

*Proof of Proposition 2:*

We clarify what we will prove: Consider any manifold  $(M, g)$  and any  $f \in C^\infty(M)$  and chose any point  $x_0 \in M$ . Then at  $x_0$  the equation (29) will hold.

In the notation of Proposition 2, we set  $Z = a_M$ . For each  $u \in U^{a_M}$  (see the statement of Proposition 2) we denote by  $C_g^u(f, \phi^{a_M})$  the complete contraction which arises from  $C_g^u(f)$  by replacing each of the  $a_M$  factors  $S\nabla_{r_1 \dots r_p}^{(p)} P_{r_{p+1} r_{p+2}}$  by  $S\nabla_{r_1 \dots r_{p+2}}^{p+2} \phi$ .

Then, by virtue of the transformation law (20) and the conformal invariance of the Weyl tensor we derive that:

$$(0 \Rightarrow) Im_\phi^{a_M | (w, w \cdot \kappa - K)} \left[ \sum_{u \in U} a_u C_g^u(f) \right] = (-1)^{a_M} \sum_{u \in U^{a_M}} a_u C_g^u(f, \phi^{a_M}) + \sum_{j \in J} a_j C_g^j(f, \phi) \quad (34)$$

here each of the contractions  $C_g^j(f, \phi)$  is in the general form:

$$contr(\nabla^{(m_1)} R \otimes \dots \otimes \nabla^{m_s} R \otimes \nabla^{(p_1)} f \otimes \dots \otimes \nabla^{(p_\kappa)} f \otimes \nabla^{(y_1)} \phi \otimes \dots \otimes \nabla^{(y_{a_M})} \phi)$$

and has at least  $\sigma + 1$  factors. Note that this equation holds for any  $\phi \in C^\infty(M)$ .

Then, we just pick a function  $\phi \in C^\infty(M)$  so that at our chosen  $x_0 \in M$  we have that for every  $p > 1$ :

$$S\nabla_{r_1 \dots r_p}^{(p)} \phi(x_0) = -S\nabla_{r_1 \dots r_{p-2}}^{p-2} P_{r_{p-1} r_p}(g)(x_0)$$

while if  $p \leq 1$  we have  $S\nabla^{(p)} \phi(x_0) = 0$ .

For this value of the function  $\phi$  it follows that (34) implies Proposition 2 when the extra restrictions are not applicable. When the extra restrictions are applicable, we must also observe that the correction terms (with more factors) that we are introducing in (34) also satisfy the extra restrictions-this follows by the same arguments as in the proof of Lemma 2 (see the appendix).  $\square$

*Proof of Proposition 3:*

We start by recalling that under the hypothesis of Proposition 3, all the complete contractions  $C_g^u(f)$  with  $\sigma$  factors that appear in the expression for  $L_g(f)$  must have  $\sigma - \kappa$  factors  $\nabla^{(m)} W$ . In other words, they will be in the form:

$$contr(\nabla_{r_1 \dots r_{m_1}}^{(m_1)} W_{ijkl} \otimes \dots \otimes \nabla_{u_1 \dots u_{m_\sigma - \kappa}}^{(m_\sigma - \kappa)} W_{i'j'k'l'} \otimes \nabla_{y_1 \dots y_{p_1}}^{(p_1)} f \otimes \dots \otimes \nabla_{t_1 \dots t_{p_\kappa}}^{(p_\kappa)} f) \quad (35)$$

For each complete contraction  $C_g^l(f)$  in the above form with no internal contractions we will construct a complete contraction in the ambient metric,

$C_{\tilde{g}^{n+2}}^l(\tilde{u}_w)$ :

$$\text{contr}(\tilde{\nabla}_{r_1 \dots r_m}^{(m_1)} \tilde{R}_{i_1 j_1 k_1 l_1} \otimes \dots \otimes \tilde{\nabla}_{v_1 \dots v_m \sigma - \kappa}^{(m_{\sigma - \kappa})} \tilde{R}_{i_s j_s k_s l_s} \otimes \tilde{\nabla}_{y_1 \dots y_{p_1}}^{(p_1)} \tilde{u}_w \otimes \dots \otimes \tilde{\nabla}_{t_1 \dots t_{p_\kappa}}^{(p_\kappa)} \tilde{u}_w) \quad (36)$$

which is obtained from  $C_g^l(f)$  by just replacing each factor  $\nabla^{(m)}W$  by a factor  $\tilde{\nabla}^{(m)}\tilde{R}$  and each factor  $\nabla^{(p)}f$  by  $\tilde{\nabla}^{(p)}\tilde{u}_w$  and then performing the same contractions (only with respect to the metric  $\tilde{g}$ ). We now show that this Weyl operator  $\sum_{u \in U_\sigma^{0,*}} a_u \tilde{C}_g^u(\tilde{u})$  satisfies the conclusion of Proposition 3.

*Proof:* In order to see this claim, let us recall some notation from subsection 4.1. We start with  $(M, g)$  and  $f \in C^\infty(M)$  and we perform the ambient metric construction picking some  $x_0 \in M$  and mapping it to  $\tilde{x}_0 = (1, x_0, 0)$  in  $\tilde{G}$ . Recall that if the coordinates of  $(M, g)$  are  $\{x^1, \dots, x^n\}$ , then there is a special coordinate system for the ambient manifold  $(\tilde{G}^{n+2}, \tilde{g})$  of the form:  $\{t = x^0, x^1, \dots, x^n, \rho = x^{n+1}\}$ .

Now, let us furthermore recall the form of the ambient metric on  $G^{n+1} \subset \tilde{G}$ . In the coordinate system  $\{x^0, \dots, x^{n+1}\}$  the ambient metric at  $\tilde{x}_0$  is of the form:

$$\tilde{g}_{ab}^{n+2} dx^a dx^b = 2dx^0 dx^{n+1} + \sum_{i,j=1}^n g_{ij} dx^i dx^j \quad (37)$$

where  $0 \leq a, b \leq n+1$  and  $1 \leq i, j \leq n$ . We denote by  $X^0, X^1, \dots, X^n, X^\infty$  the vector fields that correspond to the coordinates  $\{x^0, x^1, \dots, x^n, x^{n+1}\}$ . In view of the form (37) of the ambient metric in this coordinate system, we observe that for each pair of indices  $a, b$  in any  $\tilde{C}_g^u(\tilde{u}_w)$  that are contracting against each other, if we assign the value  $\infty$  or  $0$  to one of the indices, then we must assign the value  $0$  or  $\infty$  to the other.

Now, in order to express a complete contraction in the form (36) as a linear combination of complete contractions in the form (24), we will have to express the components of each tensor  $\tilde{\nabla}_{r_1 \dots r_m}^{(m)} \tilde{R}_{ijkl}(\tilde{g})$  and each tensor  $\tilde{\nabla}_{r_1 \dots r_p}^{(p)} \tilde{u}_w(\tilde{g})$  in terms of the tensors  $\nabla^{(m)}W(g)$ ,  $\nabla^{(p)}P(g)$  and  $\nabla^{(y)}f(g)$ .

Using the Christoffel symbols of  $\tilde{g}$  with respect to the special coordinate system we can see the following: Consider any component  $T_{r_1 \dots r_{m+4}} = \tilde{\nabla}_{r_1 \dots r_m}^{(m)} \tilde{R}_{r_{m+1} \dots r_{m+4}}(\tilde{g})$  with  $\delta$  of the indices  $r_1, \dots, r_{m+4}$  being  $\infty$ 's,  $\epsilon$  being  $0$ 's and the rest having values between 1 and  $n$ . Let us suppose that the indices that have values between 1 and  $n$  are precisely  $r_{a_1}, \dots, r_{a_q}$ . It then follows from standard computations on the ambient metric that:

$$T_{r_1 \dots r_{m+4}} = \sum_{h=0}^{\frac{m+4-\delta}{2}} F_{r_{a_1} \dots r_{a_q}}^h + \sum_{j \in J} a_j F_{r_{a_1} \dots r_{a_q}}^j \quad (38)$$

where each  $F_{r_1 \dots r_{m+4}}^h$  stands for a linear combination of tensor products in the form  $\nabla^{(m+h)}W \otimes g \otimes \dots \otimes g$ , where the factor  $\nabla^{(m+h)}W$  has  $\delta + h$  internal contractions (thus, we observe that  $\gamma[F^h] \leq \gamma[C_g^u(f)]$ ). If  $\delta = \epsilon = 0$  then  $F_{r_1 \dots r_{m+4}}^0 = \nabla_{r_1 \dots r_m}^{(m)} W_{r_{m+1} \dots r_{m+4}}$ .



The tensors  $F_{r_1 \dots r_{m+4}}^j$  stand for linear combinations of tensor products of the form  $\nabla^{(m_1)} R \otimes \dots \otimes \nabla^{(m_u)} R \otimes g \otimes \dots \otimes g$  ( $u \geq 2$ ), where each factor  $\nabla^{(m_y)} R$  satisfies  $\gamma[\nabla^{(m_y)} R] \leq \gamma[C_g^u(f)]$ .

By complete analogy, consider any component  $T_{r_1 \dots r_p} = \tilde{\nabla}_{r_1 \dots r_p}^{(p)} \tilde{u}_w(\tilde{g})$  with  $\delta$  of the indices  $r_1, \dots, r_{m+4}$  being  $\infty$ 's,  $\epsilon$  being 0's and the rest having values between 1 and  $n$ . Let us suppose that the indices that have values between 1 and  $n$  are precisely  $r_{a_1}, \dots, r_{a_q}$ . It then follows (from standard computations on the ambient metric) that:

$$T_{r_1 \dots r_p} = \sum_{h=0}^{\frac{p-\delta}{2}} F_{r_{a_1} \dots r_{a_q}}^h + \sum_{j \in J} a_j F_{r_{a_1} \dots r_{a_q}}^j \quad (39)$$

where each  $F_{r_1 \dots r_p}^h$  stands for a linear combination of tensor products in the form  $\nabla^{(p+h)} f \otimes g \otimes \dots \otimes g$ , where the factor  $\nabla^{(p+h)} f$  has  $\delta + h$  internal contractions (thus, we observe that  $\gamma[F^h] \leq \gamma[C_g^u(f)]$  and  $\beta[F^h] \leq \beta[C_g^u(f)]$ ). If  $\delta = \epsilon = 0$  then  $F_{r_1 \dots r_p}^0 = \nabla_{r_1 \dots r_p}^{(p)} f$ .

The tensors  $F_{r_1 \dots r_{m+4}}^j$  stand for linear combinations of tensor products of the form  $\nabla^{(p')} f \otimes \nabla^{(m_1)} R \otimes \dots \otimes \nabla^{(m_u)} R \otimes g \otimes \dots \otimes g$  ( $u \geq 1$ ), where we have  $\gamma[\nabla^{(p')} f] < \gamma[C_g^u(f)]$ ,  $\beta[\nabla^{(p')} f] < \beta[C_g^u(f)]$ ,  $\gamma[\nabla^{(m_y)} R] < \gamma[C_g^u(f)]$ . Therefore, replacing the factors of  $\tilde{C}_g^u(\tilde{u}_w)$  by the right hand sides of (38), (39) we derive our claim.  $\square$

## 7 Proof of Proposition 4.

*Outline:* This (somewhat technical) section is divided in three parts: In 7.1 we recall some facts about identities holding “formally” which will be needed in the proof. In 7.2 we introduce some notation to claim Proposition 7: Very roughly, we consider the sublinear combination  $\sum_{u \in U_\sigma^\mu} a_u C_g^u(f)$  in the statement of Proposition 4, and for each  $u \in U_\sigma^\mu$  we formally construct new complete contractions  $C_g^{u, \iota | i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v$  by formally replacing each factor  $\nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  by  $\nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$  (times a constant) and also replacing each *internal contraction*  $(\nabla^a, a)$  by an expression  $(\nabla^a v, a)$ . Proposition 7 then claims that at the linearized level  $\sum_{u \in U_\sigma^\mu} a_u C_g^{u, \iota | i_1 \dots i_\mu} \nabla_{i_1} v \dots \nabla_{i_\mu} v = 0$ . We then show that Proposition 7 implies Proposition 4 (via another formal operation *Weylify*[...] which formally reverses the previous construction).

In subsection 7.3 we prove Proposition 7: Briefly, the idea is to consider the sublinear combination in the equation  $e^{(w \cdot \kappa - K)\phi} L_{e^2 \phi g}(e^{w \cdot \phi}) - L_g(f) = 0$  that is linear in  $\phi$ , thus obtaining a new equation,  $Im_\phi^{1|(w, w \cdot \kappa - K)}[L_g(f)] = 0$ . We then derive (after a careful study of transformation laws under conformal rescaling) that the sublinear combination  $Im_\phi^{1|(w, w \cdot \kappa - K)*}[L_g(f)]$  of terms with  $\sigma + 1$  factors,  $\mu - 1$  internal contractions and a factor  $\nabla \phi$  arises *only* from  $\sum_{u \in U_\sigma^\mu} a_u C_g^u(f)$  by replacing an internal contraction  $(\nabla^a, a)$  by  $(\nabla^a \phi, a)$  times

a constant. We then observe that  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[L_g(f)] = 0$ , thus deriving equation (70). Lemma 4 then essentially completes the proof of Proposition 7, modulo checking that the constant referred to above is non-zero.

### 7.1 Identities holding formally vs. by substitution.

We here very briefly explain the theorem B.3 in [6] and its straightforward generalization that appears in [2]. Theorem B.3 in [6] is itself an extension of the work of Weyl in [25].

This theorem deals with complete contractions involving tensors with certain symmetries and anti-symmetries. In the case at hand, we form complete contractions of tensor products involving *symmetric tensors*  $\{T^\alpha\}_{\alpha \in A}$  (that belong to a family  $A$ ) and *linearized curvature tensors*,  $\{R\} = \{R_{ijkl}, R_{ijkl, r_1}, \dots, R_{ijkl, r_1 \dots r_m}, \dots\}$ . The latter model (at the linearized level) the symmetries and anti-symmetries of the curvature and its covariant derivatives (see [2] for more details).

We can form complete contractions in the above objects:

$$C(T, R) = \text{contr}(R_{ijkl, r_1 \dots r_m} \otimes \dots \otimes R_{i'j'k'l', t_1 \dots t_u} \otimes T_{y_1 \dots y_b}^{\alpha_1} \otimes T_{z_1 \dots z_c}^{\alpha_\tau}) \quad (40)$$

(say with  $\rho$  factors in the form  $R_{ijkl, r_1 \dots r_m}$  and  $\tau$  factors  $T_{y_1 \dots y_b}^\alpha$ ,  $b \geq 1$ ,  $\alpha \in A$ ; the contractions are taken with respect to  $\delta_{ij}$ ) and consider linear combinations thereof.

There are then two notions of an identity holding between linear combinations of such complete contractions: Following [6], we say an identity holds *by substitution* if it holds for all possible assignments of values to the tensors in  $\{T^\alpha\}_{\alpha \in A}$  and  $\{R\}$ , which satisfy the symmetry and anti-symmetry restrictions we have imposed. We say an identity holds *formally* if we can just prove it by virtue of applying the symmetries and anti-symmetries and also the distributive rule (see [2] for a precise discussion). Clearly, if an identity holds formally, it will then also hold by substitution. Theorem 2 in [2] says that the converse is also true, subject to one restriction:

**Proposition 5** *Suppose that  $C^l(T, R)$ ,  $l \in L$  are complete contractions in the form (40), and the identity:*

$$\sum_{l \in L} a_l C^l(T, R) = 0 \quad (41)$$

*holds by substitution. Moreover suppose that each  $C^l(T, R)$  satisfies  $\tau + 2\rho \leq n$  (see the notation after (40)). It then follows that (41) also holds formally.*

*Alternatively, suppose that for each  $C^l(T, R)$  above we have  $\tau + 2\rho = n + 1$ , but also that each  $C^l(T, R)$  has one factor  $T_x^{\alpha_a}$  with rank 1, this factor only appearing once in each contraction. Then (41) again holds formally*

Finally, let us recall the notion of *linearization* for complete contractions in the form (1): For any such complete contraction  $C_g^l(f)$  we define  $\text{lin}\{C_g^l(f)\}$  to stand for the complete contraction in the form (40) that is constructed out of

$C_g^l(f)$  by substituting each “genuine” curvature term  $\nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$  by a term  $R_{ijkl, r_1 \dots r_m}$  and each “genuine” covariant derivative of  $f$ ,  $\nabla_{r_1 \dots r_p}^{(p)} f$  by a symmetric tensor  $T_{r_1 \dots r_p}$ . We recall the following fact from [2]:

**Proposition 6** *Consider a linear combination  $\sum_{l \in L} a_l C_g^l(f)$  of complete contractions in the form (1), each with  $\sigma$  factors. Suppose that for any  $g, f$  we have an identity:*

$$\sum_{l \in L} a_l C_g^l(f) = \sum_{h \in H} a_h C_g^h(f) \quad (42)$$

where the RHS stands for some linear combination of complete contractions in the form (1) with at least  $\sigma + 1$  factors. Suppose that for each  $l \in L$  we have that  $\text{lin}\{C_g^l(f)\}$  satisfies  $\tau + 2\rho \leq n$ .

Then we also have an identity:

$$\sum_{l \in L} a_l \text{lin}\{C_g^l(f)\} = 0 \quad (43)$$

which holds formally.

Alternatively, if we assume (42) where each of the contractions  $C_g^l(f)$  is in the form (1) but also has an additional factor  $\nabla\phi$  (thus  $C_g^l(f)$  has  $\sigma + 1$  factors in total) and also each  $C_g^h(f)$  also has a factor  $\nabla^{(y)}\phi$  (thus  $C_g^h(f)$  has  $\sigma + 2$  factors in total) and furthermore  $\tau + 2\rho \leq n + 1$ ; then the linearized equation (43) will hold formally.

## 7.2 Decomposing $\nabla^{(m)}W$ and reducing Proposition 4 to Proposition 7.

*Convention:* For each tensor  $T = \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$ , we will call the indices  $i, j, k, l$  the *internal indices* of  $T$ .

We will now put down some well-known identities that will be useful for our discussion. We will be considering complete contractions that involve the Weyl tensor and write each such complete contraction as a linear combination of contractions involving only the curvature tensor. We do this via the formula (7). It will be useful further down to be more precise about this decomposition, when we consider  $\nabla^{(m)}W_{ijkl}$  (i.e. an iterated covariant derivative of the Weyl tensor).

Consider any tensor  $T = \nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  where each index  $r_{a_s}$  is contracting against the (derivative) index  $r_{a_s}$ , and all the other indices are free (so we have an  $(m + 4 - x)$ -tensor). By (7) it follows that:

$$\nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} = \nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} R_{ijkl} + \sum_{z \in Z^{\delta=x+1}} a_z T^z(g) + \sum_{z \in Z^{\delta=x+2}} a_z T^z(g) \quad (44)$$

where  $\sum_{z \in Z^{\delta=x+1}} a_z T^x(g)$  stands for a linear combination of tensor products of the form  $\nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} Ric_{sq} \otimes g_{vb}$  in the same free indices as  $T$ , with the feature that there are a total of  $x+1$  internal contractions in the tensor  $\nabla^{(m)} Ric_{sq}$  (including the one in the tensor  $Ric_{sq} = R^a_{saq}$  itself).  $\sum_{z \in Z^{\delta=x+2}} a_z T^x(g)$  stands for a linear combination of tensor products of the form  $\nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} S \otimes g_{vb} \otimes g_{hj}$  ( $S$  is the scalar curvature) in the same free indices as  $T$ , with the feature that there are a total of  $x+2$  internal contractions in the tensor  $\nabla^{(m)} S$  (including the two in the factor  $S = R^a_{ab}$  itself).

Now, we consider a factor  $T$  in the form  $T = \nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} W_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}}$  where again each upper index  $r_{a_v}$  is contracting against the lower index  $r_{a_v}$ , and moreover at least one of the indices  $r_{a_v}$  is contracting against one of the internal indices  $r_{m+1}, \dots, r_{m+4}$ . By applying (7) it can be seen that:

$$\begin{aligned} T &= \nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} W_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}} = \frac{n-3}{n-2} \nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} R_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}} \\ &+ \sum_{z \in Z^{\delta=x}} a_z T^z(g) + \sum_{z \in Z^{\delta=x+1}} a_z T^z(g) \end{aligned} \quad (45)$$

where  $\sum_{z \in Z^{\delta=x+1}} a_z T^x(g)$  stands for the same linear combination as before. Now  $\sum_{z \in Z^{\delta=x}} a_z T^x(g)$  only appears in the case where there are two indices  $r_{a_b}, r_{a_c}$  contracting against two internal indices  $r_a, r_b$  in  $W_{ijkl}$  (and moreover the indices  $r_{a_b}, r_{a_c}$  do not belong to the same block  $[ij], [kl]$ ). It stands for a linear combination of tensors  $\nabla^{r_{a_1} \dots r_{a_x}} \nabla_{r_1 \dots r_m}^{(m)} Ric_{ab}$  with  $x$  internal contractions, and with the extra feature that one of the indices  $r_{a_1}, \dots, r_{a_x}$  is contracting against one of the indices  $a, b$  in  $Ric_{ab}$ . In fact, to facilitate our discussion further down, we repeatedly apply the Ricci identity and the contracted second Bianchi identity  $2\nabla^a Ric_{ab} = \nabla_b S$  to re-write the linear combination  $\sum_{z \in Z^{\delta=x}} a_z T^z(g)$  above in the form:

$$\sum_{z \in Z^{\delta=x}} a_z T^z(g) = \sum_{z \in Z^{\delta=x}} a_z T^z(g) + \sum_{q \in Q} a_q T^q(g) \quad (46)$$

where each  $T^z(g)$ ,  $z \in Z^{\delta=x}$  is a factor of the form  $\nabla^{m+x} S$  ( $S$  is the scalar curvature), with a total of  $x$  internal contractions (including the two internal contractions in  $S = R^{ij}_{ij}$ ). Also,  $T^q$  are quadratic correction terms, partial contractions of the form  $pcontr(\nabla^{(b)} R \otimes \nabla^{(c)} R)$ , moreover we have  $\gamma[\nabla^{(b)} R] \leq \gamma[T]$  and  $\beta[\nabla^{(b)} R] \leq \beta[T]$  and similarly for  $\nabla^{(c)} R$  ( $T$  is the left hand side of (45)).

Now, we will prove our proposition 4 by virtue of the next proposition, for which we will need a little more notation.

For each contraction  $C_g^u(f)$  in the form (35), we define  $C_g^{u,\iota}(f)$  to stand for the complete contraction (times a constant), in the form:

$$\text{contr}(\nabla^{y_1 \dots y_b} \nabla_{r_1 \dots r_m}^{(m_1)} R_{ijkl} \otimes \dots \otimes \nabla^{w_1 \dots w_z} \nabla_{t_1 \dots t_{m_x}}^{(m_x)} R_{i'j'k'l'} \otimes \nabla_{v_1 \dots v_{p_1}}^{(p_1)} f \otimes \dots \otimes \nabla_{v_1 \dots v_{p_q}}^{(p_q)} f) \quad (47)$$

which arises from  $C_g^u(f)$  (in the form (35)) by replacing each factor  $\nabla^{u_1 \dots u_x} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  with no internal contractions involving internal indices by a factor  $\nabla^{u_1 \dots u_x} \nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$  and each factor  $\nabla^{u_1 \dots u_x} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  with at least one internal contraction involving an internal index by a factor  $\frac{n-3}{n-2} \nabla^{u_1 \dots u_x} \nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$ .

**Definition 10** For each  $u \in U_\sigma^\mu$  we consider  $C_g^{u,\iota}(f)$  in the form (47) and we construct a pair of lists  $(L_1, L_2)$ :  $L_1$  stands for the list  $(\delta_1, \dots, \delta_x)$ , where  $\delta_i$  is the number of internal contractions in the factor  $T_i (= \nabla^{(m_i)} R)$  in  $C_g^{u,\iota}(f)$ .  $L_2$  stands for the list  $(\delta_{x+1}, \dots, \delta_{x+b})$  where  $\delta_i$  is against the number of internal contractions in the factor  $T_i (= \nabla^{(p)} f)$  in  $C_g^{u,\iota}(f)$ . We then define  $RL_1, RL_2$  to stand for the decreasing rearrangements of the lists  $L_1, L_2$  after we erase the 0-entries. We call  $(RL_1, RL_2)$  the character of  $C_g^u(f)$  and denote it by  $\vec{\lambda}(u)$ . We will also denote by  $\Lambda$  the set of all pairs of lists  $\Lambda = \{\vec{\lambda}(u)\}_{u \in U_\sigma^\mu}$ .

Accordingly, we subdivide  $U_\sigma^\mu$  into subsets  $U_\sigma^{\mu, \vec{\alpha}}$ ,  $\vec{\alpha} \in \Lambda$  according to the rule that  $u \in U_\sigma^{\mu, \vec{\alpha}}$  if and only if  $C_g^u(f)$  has  $\vec{\lambda}(u) = \vec{\alpha}$ .

Then, for every  $u \in U_\sigma^\mu$ , we define  $C_g^{u,\iota|i_1 \dots i_\mu}(f)$  to stand for the tensor field that arises from  $C_g^{u,\iota}(f)$  by replacing each internal contraction  $(\nabla^a, {}_a)$  by a free index  $_a$  (in other words we erase  $\nabla^a$  and make the index  $_a$  free). We then construct the complete contractions:

$$C_g^{u,\iota|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v$$

which arise from each  $C_g^{u,\iota|i_1 \dots i_\mu}(f)$  by just contracting each of the free indices against a factor  $\nabla v$  ( $v$  is some arbitrary scalar function). The notion of character naturally extends to tensor fields, or to complete contractions in the above form, where instead of internal contractions we count free indices or the numbers of indices that contract against factors  $\nabla v$ , respectively.

We then consider the linearizations of the complete contractions above, which we denote by:

$$\text{lin}\{C_g^{u,\iota|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v\} \quad (48)$$

(we will denote by  $R_{ijkl, r_1 \dots r_m}$  the linearized curvature factor that replaces  $\nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$ , by  $\Phi_{r_1 \dots r_p}$  the symmetric tensor that replaces  $\nabla_{r_1 \dots r_p}^{(p)} f$ , and by  $v_s$  the vector that replaces  $\nabla_s v$ ).

We claim a new Proposition, which will imply our Proposition 4:

**Proposition 7** In the above equation, we claim that for each  $\vec{\alpha} \in \Lambda$ :

$$\sum_{u \in U_\sigma^{\mu, \vec{\alpha}}} a_u \text{lin}\{C_g^{u,\iota|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v\} = 0 \quad (49)$$

and moreover the above holds formally.

*Proof that Proposition 7 implies Proposition 4:*

Clearly, in order to prove Proposition 4 it would suffice to prove that for every  $\vec{\alpha} \in \Lambda$ :

$$\sum_{u \in U_{\vec{\alpha}}^{\mu}} a_u C_g^u(f) = \sum_{u \in U'} a_u C_g^u(f) + \sum_{j \in J} a_j C_g^j(f) \quad (50)$$

where the right hand side is as in the statement of Proposition 4. We will show this below. We start by putting down a few identities.

We recall that the Weyl tensor  $W_{ijkl}$  is antisymmetric in the indices  $i, j$  and also  $W_{ijkl} = W_{klij}$ . It also satisfies the first Bianchi identity. Nevertheless, it does not satisfy the second Bianchi identity. We now present certain substitutes for the second Bianchi identity:

Firstly, if the indices  $r, i, j, k, l$  are all free we have that:

$$\nabla_r W_{ijkl} + \nabla_j W_{rikl} + \nabla_i W_{jrkl} = \sum (\nabla^s W_{srtj} \otimes g) \quad (51)$$

where the symbol  $\sum (\nabla^s W_{srtj} \otimes g)$  stands for a linear combination of tensor products of the three-tensor  $\nabla^s W_{srtj}$  (i.e., essentially the Cotton tensor) with an un-contracted metric tensor. The exact form of  $\sum (\nabla^s W_{srtj} \otimes g)$  is not important so we do not write it down.

On the other hand, if the indices  $i, j, k, l$  are free we then have:

$$\nabla_s^s W_{ijkl} + \frac{n-2}{n-3} \nabla_j^s W_{sikl} + \frac{n-2}{n-3} \nabla_i^s W_{jskl} = \sum (\nabla^{st} W_{svtr} \otimes g) + Q(R) \quad (52)$$

where the symbol  $\sum (\nabla^{st} W_{svtr} \otimes g)$  stands for a linear combination of tensor products:  $\nabla^{st} W_{svtr} \otimes g_{ab}$  ( $g_{ab}$  is an un-contracted metric tensor-note that there are *two* internal contractions in the factor  $\nabla^{ik} W_{ijkl}$ ).  $Q(R)$  stands for a quadratic expression in the curvature tensor (without covariant derivatives). Again the exact form of these expressions is not important so we do not write them down.

Whereas, if the indices  $r, i, j, l$  are free we will then have that:

$$\nabla^k \nabla_r W_{ijkl} + \nabla^k \nabla_j W_{rikl} + \nabla^k \nabla_i W_{jrkl} = Q(R) \quad (53)$$

Finally:

$$\nabla^{ri} \nabla_r W_{ijkl} + \nabla^{ri} \nabla_j W_{rikl} + \nabla^{ri} \nabla_i W_{jrkl} = Q(R) \quad (54)$$

Of course, if we take covariant derivatives of these equations, they continue to hold. We will collectively call these identities the “fake” second Bianchi identities.

A lemma that will be useful in the more technical parts of this proof is the following:

**Lemma 2** Consider any complete contraction  $C_g(f)$  in either of the forms (1), (24). Suppose we apply either the identity (18), or any of the fake second Bianchi identities, and thus write:  $C_g(f) = \sum_{k \in K} a_k C_g^k(f)$ .

We then claim that if  $C_g(f)$  satisfies the extra restrictions then so does each  $C_g^k(f)$ .

We will prove this lemma in the appendix. Let us now make a few easy observations.

Any complete contraction  $C_g^u(f)$  in the form (35) that has two antisymmetric indices  $i, j$  or  $k, l$  in a given factor  $T = \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  contracting against two derivatives in the same factor can be written:  $C_g^u(f) = \sum_{r \in R} a_r C_g^r(f)$  each  $C_g^r(f)$  with  $\sigma + 1$  factors. This is straightforward from the antisymmetry of the indices  $i, j$  and  $k, l$  and the Ricci identity. Moreover we have that each  $C_g^r(f)$  will satisfy the extra restrictions when they are applicable (by Lemma 2). Thus we may prove our Proposition under the extra assumption that all complete contractions  $C_g^u(f)$  have no factor  $T = \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  with two antisymmetric indices  $i, j$  or  $k, l$  contracting against two derivative indices in  $T$ . Moreover, since the Weyl tensor is trace-free we may assume that each  $C_g^u(f)$ ,  $u \in U_\sigma$  has no factor  $\nabla^{(m)} W$  with two internal indices contracting between themselves.

Now we will show that Proposition 7 implies Proposition 4:

**Definition 11** We define an operation *Weylify* that acts on linearized complete contractions in the form (48) as follows: We identify the indices in each linearized factor  $R_{ijkl, r_1 \dots r_m}$  and  $\Phi_{w_1 \dots w_a}$  that are contracting against factors  $v_s$ . Then, we pick out each factor  $\Phi_{w_1 \dots w_a}$  where the indices  $w_{h_1}, \dots, w_{h_b}$  are contracting against factors  $v_s$  and replace it by a factor  $\nabla^{w_{h_1} \dots w_{h_b}} \nabla_{w_1 \dots w_a}^a f$  (and we also erase the factors  $v_s$ ), thus obtaining a factor with  $h_b$  internal contractions.

Moreover, we pick out each factor  $R_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}, r_1 \dots r_m}$  and we identify the indices  $r_{h_1}, \dots, r_{h_b}$  that are contracting against factors  $v_s$ . We then inquire whether any of the internal indices  $r_{m+1}, \dots, r_{m+4}$  are contracting against a factor  $v_s$ . If not, and we replace the factor  $R_{ijkl, r_1 \dots r_m}$  by a factor  $\nabla^{r_{h_1} \dots r_{h_b}} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$ . If there are internal indices contracting against factors  $v_s$  we then replace  $R_{r_{m+1} r_{m+2} r_{m+3} r_{m+4}, r_1 \dots r_m}$  by a factor  $\frac{n-2}{n-3} \nabla^{r_{h_1} \dots r_{h_b}} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$ . This operation *Weylify*[...] extends to linear combinations.

By definition, we observe that:

$$\text{Weylify}\left\{ \sum_{u \in U_\sigma^\mu, \bar{\alpha}} a_u \text{lin}\{C_g^{u, \iota|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v\} \right\} = \sum_{u \in U_\sigma^\mu, \bar{\alpha}} a_u C_g^u(f) \quad (55)$$

By virtue of the above and also of equations (51), (52), (53), (54), we observe that if we *repeat* the sequence of permutations by which we make the left hand side of (49) formally zero, we can make  $\sum_{u \in U_\sigma^\mu, \bar{\alpha}} a_u C_g^u(f)$  formally zero, modulo

introducing correction terms by virtue of the right hand sides of the equations (51), (52), (53), (54) and also by virtue of the Ricci identity when we interchange adjacent derivative indices.

But by inspection of the formulas (51), (52), (53), (54) and also by using Lemma 2, it follows that the correction terms that arise thus are in the form:

$$\sum_{u \in U'} a_u C_g^u(f) + \sum_{j \in J} a_j C_g^j(f)$$

(in the notation of Proposition 4)).  $\square$

### 7.3 Proof of Proposition 7.

Our point of departure in this proof is the equation:

$$Im_\phi^{1|(w, w \cdot \kappa - K)}[L_g(f)] = 0.$$

In order to apply our arguments, we will be writing out  $Im_\phi^{1|(w, w \cdot \kappa - K)}[L_g(f)]$  as a linear combination of complete contractions in the form:

$$contr(\nabla^{(m_1)} R \otimes \dots \otimes \nabla^{(m_t)} R \otimes \nabla^{(p)} \phi) \quad (56)$$

We straightforwardly observe that under the hypotheses of Proposition 4:

$$(0 =) Im_\phi^{1|(w, w \cdot \kappa - K)}[L_g(f)] = \sum_{h \in H_{\sigma+1}} a_h C_g^h(f, \phi) + \sum_{h \in H_{\geq \sigma+2}} a_h C_g^h(f, \phi) \quad (57)$$

Here the complete contractions indexed in  $H_{\sigma+1}$  have  $\sigma + 1$  factors while the ones indexed in  $H_{\geq \sigma+2}$  have at least  $\sigma + 2$  factors. All contractions on the RHS are understood to be in the form (56); this can be done by decomposing the Weyl tensor. We observe that by virtue of the transformation laws (19), (21) and (20) and by virtue of the fact that the complete contractions in  $L_g(f)$  have weight  $-K$ , all complete contractions in (57) will have a factor  $\nabla^{(p)} \phi$  with  $p \geq 1$ . Now, we denote by  $H_{\sigma+1}^* \subset H_{\sigma+1}$  the index set of complete contractions with a factor  $\nabla \phi$  (with only one derivative). We observe, by virtue of the transformation laws (19), (21) and (20), that this sublinear combination can *only* arise from the complete contractions of length  $\sigma$  in  $L_g(f)$  by applying the transformation law (21) *or* by bringing out a factor  $\nabla \phi$  by virtue of the transformations  $W_{ijkl} \rightarrow e^{2\phi} W_{ijkl}$  and  $f \rightarrow e^{w\phi} f$  (when  $w \neq 0$ ).

More is true: Consider  $Im_\phi^{1|(w, w \cdot \kappa - K)}[C_g^u(f)]$ , where  $u \in U_\sigma^\mu$ . We pick out the sublinear combination of complete contractions (*in the form (56)*) with a factor  $\nabla \phi$  and with  $\mu - 1$  internal contractions in total. For each  $u \in U_\sigma^\mu$  we denote this sublinear combination by  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$ .

It is straightforward to observe that for each  $u \in U_\sigma^\mu$ , any complete contraction in  $Im_\phi^{1|(w, w \cdot \kappa - K)}[C_g^u(f)]$  (which is written as a linear combination of complete contractions in the form (56)) that does not belong to the sublinear



combination  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  must either have a factor  $\nabla^{(p)}\phi$  with  $p > 1$  or must have at least  $\mu$  internal contractions.

Furthermore, it is equally straightforward that for any complete contraction  $C_g^u(f)$ ,  $u \in U_\sigma \setminus U_\sigma^\mu$ , ( $C_g^u(f)$  by hypothesis is in the form (35) with  $\sigma$  factors and at least  $\mu + 1$  internal contractions),  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  must consist of complete contractions in the form (56) with either a factor  $\nabla^{(p)}\phi$ ,  $p > 1$ , or with at least  $\mu$  internal contractions.

Therefore, by the above discussion, we can re-express (57) to obtain:

$$\sum_{u \in U_\sigma^\mu} a_u Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)] + \sum_{v \in V_1} a_v C_g^v(f, \phi) + \sum_{v \in V_2} a_v C_g^v(f, \phi) = 0 \quad (58)$$

where the complete contractions indexed in  $V_1$  are in the form (56) and have  $\sigma + 1$  factors and a factor  $\nabla^{(p)}\phi$ , and moreover either have  $p > 1$  or have  $p = 1$  and at least  $\mu$  internal contractions (we accordingly divide  $V_1$  into  $V_1^\alpha, V_1^\beta$ ). The complete contractions indexed in  $V_2$  are generic in the form (56) and have at least  $\sigma + 2$  factors. Now, since the above holds at any point  $x_0 \in M$  and for any values we assign to the jet of  $\phi$  at  $x_0$ , we derive that:

$$\sum_{u \in U_\sigma^\mu} a_u Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)] + \sum_{v \in V_1^\beta} a_v C_g^v(f, \phi) + \sum_{v \in V_2'} a_v C_g^v(f, \phi) = 0 \quad (59)$$

(where the contractions indexed in  $V_2'$  have at least  $\sigma + 2$  factors).

By Proposition 6, the above must hold formally for the *linearizations* of the complete contractions with  $\sigma + 1$  factors (the degree of the complete contractions will be at most  $n + 1$ —see the note after definition 6). In the notation of Proposition 6, we will have:

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]\} + \sum_{v \in V_1^\beta} a_v \text{lin}\{C_g^v(f, \phi)\} = 0 \quad (60)$$

Now, we only have to observe that the total number of internal contractions in each of the linearized complete contractions above remain invariant under the permutations by which we make the above formally zero. Therefore, (60) implies:

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]\} = 0 \quad (61)$$

and moreover the above still holds formally.

We will use the above equation to prove our Proposition 7, but in order to do this we must better understand how the sublinear combination  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  arises from  $C_g^u(f)$ ,  $u \in U_\sigma^\mu$ .

In order to understand  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$ , for any  $u \in U_\sigma^\mu$ , we will first seek to separately understand the transformation of each factor  $T$  in  $C_g^u(f)$  under the operation  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[\dots]$ . If  $T = \nabla^{(m)}W$ , we define  $Im_\phi^1[T]$  to stand for the terms of homogeneity one in  $\phi$  in  $\nabla^{(m)}W(e^{2\phi}g)$ , which have exactly one derivative on  $\phi$ ; the terms in  $Im_\phi^1[T]$  will be in the form  $\nabla^{(m-1)}W \otimes \nabla\phi$  or  $\nabla^{(m-1)}W \otimes \nabla\phi \otimes g$ , i.e. we are not decomposing the Weyl tensor. If  $T_y = \nabla^{(p)}f$ , we define  $Im_\phi^1[T_y]$  to stand for the terms of homogeneity one in  $\phi$  in  $[\nabla^{(p)}(e^{w \cdot \phi}f)](e^{2\phi}g)$ , which have exactly one derivative on  $\phi$ .

It follows that the sublinear combination of terms in  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  with a factor  $\nabla\phi$  arises by picking out each factor  $T$  in  $C_g^u(f)$ , replacing it by  $Im_\phi^1[T]$  and then summing over all these possible replacements. Now, consider any factor  $T_y$  in  $C_g^u(f)$  which has  $\delta > 0$  internal contraction in  $C_g^u(f)$ ; define  $R[T_y]$  to stand for the sublinear combination of terms in  $Im_\phi^1[T_y]$  in the form  $\nabla^{(m-1)}W$  or  $\nabla^{(p-1)}f$  with exactly  $\delta - 1$  internal contractions. Define  $Im_\phi^{1|(w, w \cdot \kappa - K)|\#}[C_g^u(f)]$  to stand for the sublinear combination in  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  which arises by picking out each factor  $T_y$  in  $C_g^u(f)$ , replacing it by  $R[T_y]$  and then summing over all these substitutions; then observe that  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$  will be a *sublinear combination* of  $Im_\phi^{1|(w, w \cdot \kappa - K)|\#}[C_g^u(f)]$ . In view of this, we set out to understand the form of  $R[T_y]$  for each factor  $T_y$  in  $C_g^u(f)$ .

We will introduce some notation to do this. Consider any  $C_g^u(f)$ ,  $u \in U_\sigma^\mu$  in the form (35). We pick out the factors  $T_y$  that contain an internal contraction, and assume they are indexed in the set  $\{T_1, \dots, T_{x_u}, T_{x_u+1}, \dots, T_{x_u+b_u}\}$ . We make the convention that the factors  $\{T_1, \dots, T_{x_u}\}$  are in the form  $\nabla^{(m)}W$ , while the factors  $T_{x_u+1}, \dots, T_{x_u+b_u}$  are in the form  $\nabla^{(p)}f$ .

For convenience, we will repeatedly apply the Ricci identity to write each complete contraction  $C_g^u(f)$ ,  $u \in U_\sigma$  in the form:

$$\begin{aligned} & \text{contr}(\nabla^{y_1 \dots y_w} \nabla_{r_1 \dots r_m}^{(m_1)} W_{ijkl} \otimes \dots \otimes \nabla^{h_1 \dots h_b} \nabla_{c_1 \dots c_{m_\sigma - \kappa}}^{(m_{\sigma - \kappa})} W_{i'j'k'l'}) \\ & \nabla^{f_1 \dots f_z} \nabla_{q_1 \dots q_{p_1}}^{(p_1)} f \otimes \dots \otimes \nabla^{d_1 \dots d_x} \nabla_{i_1 \dots i_{p_\kappa}}^{(p_\kappa)} f \end{aligned} \quad (62)$$

where we are making the convention that the raised indices are contracting against one of the lower indices in the same factor, and all the lower indices that are not contracting against a raised index in the same factor are contracting against some lower index in another factor. By Lemma 2, we observe that the correction terms we obtain will satisfy the extra restrictions when they are relevant.

We first consider any factor  $T_y$ ,  $y \in \{x_u + 1, \dots, x_u + b_u\}$  that has, say,  $\delta$  internal contractions. Recall  $T_y$  in the form  $T_y = \nabla^{u_1 \dots u_\delta} \nabla_{r_1 \dots r_p}^{(p)} f$  with the conventions that each of the indices  $u_1, \dots, u_\delta$  is contracting against one of the indices  $r_1, \dots, r_p$ . Moreover each of the indices  $r_1, \dots, r_p$  that is not contracting against an index  $u_1, \dots, u_\delta$  is a free index. It follows that (modulo terms with three factors)  $R[T_y]$  will be:

$$R[T_y] = [\delta \cdot (n - 2) - 4 \binom{\delta}{2} + 2\delta w](\nabla)^{u_1} \phi \nabla^{u_2 \dots u_\delta} \nabla_{r_1 \dots r_p}^{(p)} f \quad (63)$$

Now, we pick out any factor  $T_y, y \in \{1, \dots, x_u\}$ , in the form  $\nabla^{u_1 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$ . In order to understand  $R[T_y]$  in this setting we must distinguish three cases: In the first case we have that all the internal contractions in  $T_y$  are between derivative indices. In the second case we have that there is precisely one internal contraction involving an internal index (and with no loss of generality we will assume that  $^{u_1}$  is contracting against the index  $_i$  in  $W_{ijkl}$ ). In the third case we have there are precisely two internal contractions involving internal indices, (and with no loss of generality we will assume that the indices  $^{u_1}, ^{u_2}$  are contracting against the indices  $_i, _k$  respectively).

In the case where all the internal contractions in  $T_y$  are between derivative indices we see that (modulo terms with 3 factors)  $R[T_y]$  will be:

$$R[T_y] = [\delta \cdot (n - 2) - 4 \binom{\delta}{2}] \nabla^{u_2 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^{u_1} \phi \quad (64)$$

In the case where there is precisely one internal contraction in  $T_y$  (between the indices  $^{u_1}, _i$ ), we also assume for convenience that if  $\delta > 1$  then  $^{u_2}$  is contracting against the index  $_{r_1}$ . Then, (modulo terms with 3 factors)  $R[T_y]$  will be the sum:

$$\begin{aligned} & (n - 3) \nabla^{u_2 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^i \phi + [(\delta - 1)(n - 2) - 4 \binom{\delta}{2}] \nabla^{u_1 u_3 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^{u_2} \phi + \\ & (\delta - 1) \nabla^{u_1 u_3 \dots u_\delta} \nabla_{jr_2 \dots r_m}^{(m)} W_{ir_1 kl} (\nabla)^{r_1} \phi \end{aligned} \quad (65)$$

Now, we consider the case where a factor  $T_i$  is in the form  $\nabla^{u_1 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  where two of the internal indices ( $i$  and  $k$ ) are involved in an internal contraction. We then assume for convenience that if  $\delta > 2$  then  $^{u_3}$  is contracting against  $_{r_1}$ . Thus, modulo terms with 3 factors,  $R[T_i]$  will be:

$$\begin{aligned} & (n - 4) \nabla^{u_2 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^i \phi + (n - 4) \nabla^{u_1 u_3 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^k \phi + \\ & (\delta - 2)(n - 2) \nabla^{u_1 u_2 u_4 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^{r_1} \phi \\ & - 4 \left[ \binom{\delta - 2}{2} + 2(\delta - 2) \right] \nabla^{u_1 u_2 u_4 \dots u_\delta} \nabla_{r_1 \dots r_m}^{(m)} W_{ijkl} (\nabla)^{r_1} \phi + \\ & (\delta - 2) \nabla^{u_1 u_2 u_4 \dots u_\delta} \nabla_{jr_2 \dots r_m}^{(m)} W_{ir_1 kl} (\nabla)^{r_1} \phi + (\delta - 2) \nabla^{u_1 u_2 u_4 \dots u_\delta} \nabla_{lr_2 \dots r_m}^{(m)} W_{ijkl} (\nabla)^{r_1} \phi \end{aligned} \quad (66)$$

Now, for each  $y \in \{1, \dots, x_u + b_u\}$  we define  $C_g^{u,y}(f, \phi)$  to be the complete contraction (or linear combination of complete contractions), in the form (56), that arises from  $C_g^u(f)$  by replacing the factor  $T_y$  by  $R[T_y]$  and then replacing

each factor  $\nabla_{r_1 \dots r_m}^{(m)} W_{ijkl}$  by a factor  $\nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$  or  $\frac{n-3}{n-2} \nabla_{r_1 \dots r_m}^{(m)} R_{ijkl}$  depending on whether there is no internal contraction involving an internal index or not.

Then, for each  $u \in U_\sigma^\mu$  we define  $C_g^{u,+}(f, \phi)$  to stand for the linear combination of complete contractions:

$$C_g^{u,+}(f, \phi) = \sum_{y=1}^{x_u + b_u} C_g^{u,y}(f, \phi) \quad (67)$$

Now, by the definitions of  $R[T_y]$  and  $Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)]$ , by the formulas above and by the decomposition formulas for  $\nabla^{(m)} W$ , we derive that if  $C_g^u(f)$ ,  $u \in U_\sigma^\mu$ , has no factor  $\nabla^{(m)} W$  with two internal contractions involving internal indices then:

$$Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)] = C_g^{u,+}(f, \phi) \quad (68)$$

On the other hand, if  $C_g^u(f)$  does contain factors  $\nabla^{(m)} W$  with two internal contractions involving internal indices then:

$$Im_\phi^{1|(w, w \cdot \kappa - K)|*}[C_g^u(f)] = C_g^{u,+}(f, \phi) + \sum_{d \in D} a_d C_g^d(f, \phi) \quad (69)$$

where  $\sum_{d \in D} a_d C_g^d(f)$  stands for a generic linear combination of complete contractions in the form (56) with the extra feature that they contain  $a > 0$  factors  $\nabla_{r_1 \dots r_z}^z S$ , where  $S$  is the scalar curvature. The contractions  $C_g^d(f, \phi)$  arise by virtue of the factors  $T^z$ ,  $z \in Z^{\delta=x}$  in (45).

Thus, equation (61) implies:

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{C_g^{u,+}(f, \phi)\} + \sum_{d \in D} a_d \text{lin}\{C_g^d(f, \phi)\} = 0 \quad (70)$$

We then have two claims.

**Lemma 3** *Given (70) we claim that:*

$$\sum_{d \in D} a_d \text{lin}\{C_g^d(f, \phi)\} = 0 \quad (71)$$

Observe that if we can show the above, we will then have:

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{C_g^{u,+}(f, \phi)\} = 0 \quad (72)$$

We present our next claim with a little more notation: We denote by  $\text{lin}\{C_g^{u,+,i_1 \dots i_{\mu-1}}(f, \phi)\}$  the (linearized) tensor field that arises from  $\text{lin}\{C_g^{u,+}(f, \phi)\}$  by replacing each of the internal contractions by a free index (meaning that in each complete contraction  $(\nabla^a, a)$  we erase  $\nabla^a$  and make the index  $a$  free). We then form a (linearized) complete contraction  $\text{lin}\{C_g^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v\}$ . Our second claim is that:

**Lemma 4** *Assuming (72) and employing the notation above we have that:*

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{C_g^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v\} = 0 \quad (73)$$

and moreover the above holds formally.

We will show these two Lemmas below. For now, let us check how these Lemmas will imply Proposition 7 (and hence Proposition 4). We arbitrarily pick out an element  $\vec{\alpha} \in \Lambda$  (see definition 10) and we will show Proposition 7 for the index set  $U_\sigma^{\mu, \vec{\alpha}}$ . We must distinguish two cases: Either for the character  $\vec{\alpha} = (RL_1 | RL_2)$  we have  $RL_1 = \emptyset$  (in which case we must also necessarily have  $RL_2 \neq \emptyset$ , since  $\mu > 0$ ), or we have  $RL_1 \neq \emptyset$ . For future reference, we will write out  $\vec{\alpha}$  explicitly:

$$\vec{\alpha} = (RL_1 | RL_2) = (\zeta_1, \dots, \zeta_a | \xi_1, \dots, \xi_b) \quad (74)$$

Now, we first consider the case  $RL_1 = \emptyset$  for our chosen  $\vec{\alpha} \in \Lambda$ . As we observed, this implies that  $RL_2 \neq \emptyset$ , i.e. for each  $u \in U_\sigma^{\mu, \vec{\alpha}}$  we have  $\xi_1 > 0$ .

We then consider the sublinear combination in (73) which contains complete contractions with the (linearized) factor  $\nabla \phi$  contracting against a (linearized) factor  $T = \text{lin}\{\nabla^{(p)} f\}$ , and moreover the factor  $T$  is also contracting against precisely  $\xi_1 - 1$  factors  $\nabla v$ . If we denote the linear combination of such complete contractions by  $\text{lin}\{S_g(f, \phi, v)\}$  then since (73) holds formally, we derive that:

$$\text{lin}\{S_g(f, \phi, v)\} = 0 \quad (75)$$

and moreover the above holds formally. Now, we set  $\phi = v$  in the above. By applying equations (61) and (67) we observe that:

$$\begin{aligned} (0 =) \text{lin}\{S_g(f, v, v)\} &= \text{Const}(\vec{\alpha}) \sum_{u \in U_\sigma^{\mu, \vec{\alpha}}} a_u \text{lin}\{C_g^{u|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v\} \\ &+ \sum_{s \in S} a_s \text{lin}\{C_g^{s, i_1 \dots i_\mu} \nabla_{i_1} v \dots \nabla_{i_\mu} v\} \end{aligned} \quad (76)$$

( $C_g^{u|i_1 \dots i_\mu}$  stands for the tensor field that arises from  $C_g^u(f)$  by making all complete contractions into free indices). Here each  $\text{lin}\{C_g^{s, i_1 \dots i_\mu} \nabla_{i_1} v \dots \nabla_{i_\mu} v\}$  has a character that is *different* from our chosen  $\vec{\alpha}$ . Therefore, since the above holds formally and since the character of the complete contractions remains unaltered under the permutations that make (76) *formally* zero, we derive:

$$\text{Const}(\vec{\alpha}) \sum_{u \in U_\sigma^{\mu, \vec{\alpha}}} a_u \text{lin}\{C_g^{u|i_1 \dots i_\mu}(f) \nabla_{i_1} v \dots \nabla_{i_\mu} v\} = 0 \quad (77)$$

The constant  $\text{Const}(\vec{\alpha})$  only depends on our chosen character  $\vec{\alpha}$ . To be precise, if we denote by  $m$  the number of times that  $\xi_1$  appears in  $\vec{\alpha}$  ( $m \geq 1$ ), by applying (63) and the definitions, it follows that:

$$Const(\vec{\alpha}) = m \cdot [\xi_1 \cdot (n-2) - 4 \binom{\xi_1}{2} + 2\xi_1 \cdot w] = m \cdot \xi_1 \cdot [n - 2\xi_1 + 2w]$$

and we observe that  $Const'(\vec{\alpha}) \neq 0$  if  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ . Furthermore, if  $w = -\frac{n}{2} + k$  for some  $k \in \mathbb{Z}_+$ , we will still have  $Const(\vec{\alpha}) \neq 0$  by our extra restriction on  $\beta[L_g(f)]$ , which ensures that  $\xi_1 < k$ .

Now, we proceed with the second case, where  $RL_1 \neq \emptyset$ . Thus, for every  $u \in U_{\sigma}^{\mu, \vec{\alpha}}$  there is an internal contraction in some factor  $\nabla^{(m)}W$ . Now, similarly to the case above, we will consider the sublinear combination of contractions in (73) for which  $\nabla\phi$  is contracting against a linearized curvature factor, say  $T$ , and also  $T$  is contracting against precisely  $(\zeta_1 - 1)$  factors  $\nabla v$ .

If we denote the linear combination of such complete contractions by  $lin\{S'_g(f, \phi, v)\}$  then since (73) holds formally, we derive that:

$$lin\{S'_g(f, \phi, v)\} = 0 \quad (78)$$

and moreover the above holds formally. Now, we set  $\phi = v$  and derive a new equation which we denote:  $lin\{S'_g(f, v, v)\} = 0$  (and the above still holds formally). By the same argument as for the previous case, we derive that:

$$\begin{aligned} (0 =) lin\{S'_g(f, \phi, v)\} &= Const'(\vec{\alpha}) \sum_{u \in U_{\sigma}^{\mu, \vec{\alpha}}} a_u lin\{C_g^{u|i_1 \dots i_{\mu}}(f) \nabla_{i_1} v \dots \nabla_{i_{\mu}} v\} \\ &+ \sum_{s \in S} a_s lin\{C_g^{s, i_1 \dots i_{\mu}} \nabla_{i_1} v \dots \nabla_{i_{\mu}} v\} \end{aligned} \quad (79)$$

(here  $C_g^{u|i_1 \dots i_{\mu}}$  stands for the tensor field that arises from  $C_g^u(f)$  by making all internal contractions into free indices). Here again the contractions  $lin\{C_g^{s, i_1 \dots i_{\mu}} \nabla_{i_1} v \dots \nabla_{i_{\mu}} v\}$  have a *different* character from our chosen  $\vec{\alpha}$ . Therefore, since the above holds formally, we derive:

$$Const'(\vec{\alpha}) \sum_{u \in U_{\sigma}^{\mu, \vec{\alpha}}} a_u lin\{C_g^{u|i_1 \dots i_{\mu}}(f) \nabla_{i_1} v \dots \nabla_{i_{\mu}} v\} = 0 \quad (80)$$

The constant  $Const'(\vec{\alpha})$  only depends on our chosen character  $\vec{\alpha}$ . In particular, if we denote by  $f$  the number of times that the number  $\zeta_1$  appears in (74), then it follows that:

$$Const'(\vec{\alpha}) = f \cdot [\zeta_1 \cdot (n-2) - 4 \binom{\zeta_1}{2}] = f \cdot \zeta_1 \cdot (n - 2\zeta_1) \quad (81)$$

(81) follows from (63), (64), (65), (66), (67). Now, in the case where  $n$  is odd, we clearly see that  $Const'(\vec{\alpha}) \neq 0$ . In  $n$  is even, we have only have to recall that  $\zeta_1$  corresponds to the number of internal contractions in some factor  $T = \nabla^{(m)}W$

in some complete contraction in  $L_g^\sigma(f)$ . But then our restriction  $\gamma[L_g(f)] < \frac{n}{2}$  guarantees that  $\zeta_1 < \frac{n}{2}$  (observe that  $\gamma[T] \geq \zeta_1$ ). Thus we derive  $Const'(\vec{\alpha}) \neq 0$  in this case also.

In view of the above we also derive our claim in the case where  $RL_1 \neq \emptyset$  for our chosen  $\vec{\alpha} \in \Lambda$ .  $\square$

In view of the above, it suffices to show our Lemmas 3 and 4 to derive Proposition 7 (and hence theorems 1, 2, 3). We will start by proving Lemma 4, because Lemma 3 is slightly more complicated.

*Proof of Lemma 4:*

In order to see this claim it will be more useful to consider complete contractions in the curvature and its covariant derivatives, i.e. in the form (56), rather than contractions in linearized tensors.

Our point of departure is equation (72). We “memorize” the sequence of permutations that make the left hand side of (72) formally zero. Now, we consider the complete contractions  $C_{g^N}^{u,+}(f, \phi)$  in any higher dimension  $N$ , so we obtain complete contractions  $C_{g^N}^{u,+}(f, \phi)$ ,  $u \in U_\sigma^\mu$ . Now, we repeat the sequence of permutation that made (57) formally zero to the linear combination  $\sum_{u \in U_\sigma^\mu} a_u C_{g^N}^{u,+}(f, \phi)$  and we derive a new equation:

$$\sum_{u \in U_\sigma^\mu} a_u C_{g^N}^{u,+}(f, \phi) = \sum_{p \in P} a_p C_{g^N}^p(f, \phi) \quad (82)$$

for any  $N \geq n$ . Here the complete contractions  $C_{g^N}^p(f, \phi)$  are in the form (56) and have at least  $\sigma + 2$  factors. The contractions  $C_{g^N}^p(f, \phi)$  arise as correction terms in (82) due to the right hand side of (18). Therefore, we derive that each  $C_g^p(f, \phi)$  will not contain a factor  $S$  (of the scalar curvature), because such factor cannot arise in correction terms appearing by virtue of the curvature identity.

One more piece of notation. For any complete contraction  $C_{g^N}(f)$  (in any dimension  $N$ ) of weight  $-K$  we define the  $l^{th}$  conformal variation  $Var_v^l$  ( $l \in \mathbb{Z}_+$  and  $v$  is an arbitrary scalar function):

$$Var_v^l[C_{g^N}(f, \phi)] = \frac{\partial^l}{\partial \lambda^l} \Big|_{\lambda=0} [C_{e^{2\lambda v} g^N}(f, \phi)]$$

We now consider  $Var_v^{\mu-1}$  of (82). Clearly, we have that:

$$Var_v^{\mu-1} \left\{ \sum_{u \in U_\sigma^\mu} a_u C_{g^N}^{u,+}(f, \phi) \right\} = Var_v^{\mu-1} \left\{ \sum_{p \in P} a_p C_{g^N}^p(f, \phi) \right\} \quad (83)$$

Now, by a careful study of the transformation laws (21) and (19) we observe that we can write:

$$\begin{aligned}
Var_v^{\mu-1} \{ \sum_{u \in U_\sigma^\mu} a_u C_{g_N}^{u,+}(f, \phi) \} &= N^{\mu-1} \sum_{u \in U_\sigma^\mu} a_u C_{g_N}^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v + \\
N^{\mu-1} \sum_{t \in T} a_t C_{g_N}^t(f, \phi, v) &+ \sum_{y=0}^{\mu-2} N^y \sum_{h \in H^y} a_h C_{g_N}^h(f, \phi, v)
\end{aligned} \tag{84}$$

All the complete contractions above are in the form:

$$\begin{aligned}
&contr(\nabla^{(m_1)} R \otimes \dots \otimes \nabla^{(m_t)} R \otimes \\
&\nabla^{(\nu_1)} f \otimes \dots \otimes \nabla^{(\nu_\kappa)} f \otimes \nabla^{(p_1)} v \otimes \dots \otimes \nabla^{(p_{\mu-1})} v)
\end{aligned} \tag{85}$$

here the complete contractions  $C_g^t(f, \phi, v)$  have  $\sigma + \mu$  factors, and they are in the form (85) and moreover have the property that at least one  $p_i, i = 1, \dots, \mu - 1$  is  $\geq 2$ .  $\sum_{h \in H^y} \dots$  stands for a generic linear combination of complete contractions in the above form (what is important is that it is multiplied by  $N^\beta$  with  $\beta < \mu - 1$ ). Furthermore, in the notation above, we also have:

$$\begin{aligned}
Var_v^{\mu-1} \{ \sum_{p \in P} a_p C_{g_N}^p(f, \phi) \} &= \\
N^{\mu-1} \sum_{t \in T'} a_t C_{g_N}^t(f, \phi, v) &+ \sum_{y=0}^{\mu-2} N^y \sum_{h \in H^y} a_h C_{g_N}^h(f, \phi, v)
\end{aligned} \tag{86}$$

where the complete contractions indexed in  $T'$  are in the form (85) but with at least  $\sigma + \mu + 1$  factors.

Therefore, by virtue of (83) (which holds formally, for  $N$  large enough) and the analysis above, we obtain:

$$\begin{aligned}
N^{\mu-1} \sum_{u \in U_\sigma^\mu} a_u C_{g_N}^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v + \\
N^{\mu-1} \sum_{t \in T \cup T'} a_t C_{g_N}^t(f, \phi, v) &+ \sum_{y=0}^{\mu-2} N^y \sum_{h \in H^y} a_h C_{g_N}^h(f, \phi, v) = 0
\end{aligned} \tag{87}$$

and furthermore this equation holds formally for  $N \geq n + \mu$ . Therefore, by just picking  $M^N = M^{n+\mu} \times S^1 \times \dots \times S^1$  we obtain:

$$\begin{aligned}
N^{\mu-1} \sum_{u \in U_\sigma^\mu} a_u C_{g^{n+\mu}}^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v + \\
N^{\mu-1} \sum_{t \in T \cup T'} a_t C_{g^{n+\mu}}^t(f, \phi, v) &+ \sum_{y=0}^{\mu-2} N^y \sum_{h \in H^y} a_h C_{g^{n+\mu}}^h(f, \phi, v) = 0
\end{aligned} \tag{88}$$



and the above still holds formally. Here  $N$  can be any integer with  $N \geq n + \mu$ . Thus, if we treat the above as a polynomial in  $N$ , we derive that:

$$\sum_{u \in U_\sigma^\mu} a_u C_{g^{n+\mu}}^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v + \sum_{t \in T \cup T'} a_t C_{g^{n+\mu}}^t(f, \phi, v) = 0 \quad (89)$$

and the above holds formally. Therefore, it must also hold formally at the linearized level, so we derive:

$$\sum_{u \in U_\sigma^\mu} a_u \text{lin}\{C_g^{u,+,i_1 \dots i_{\mu-1}}(f, \phi) \nabla_{i_1} v \dots \nabla_{i_{\mu-1}} v\} = 0 \quad (90)$$

This is precisely our desired conclusion.  $\square$

*Proof of Lemma 3:* We divide the index set  $D$  into subsets  $D^a, a = 1, \dots, \sigma - \kappa$ , according to the number of factors  $\nabla^{(a)} S$  that the contractions  $C_g^d(f), d \in D$  contains. It would clearly suffice to show that for each  $c = 1, \dots, \sigma - \kappa$  we must have:

$$\sum_{d \in D^c} a_d \text{lin}\{C_g^d(f, \phi)\} = 0 \quad (91)$$

We show (91) by a downward induction on  $c$ : We assume that for some number  $M \leq \sigma - \kappa$  we have that for every  $c > M$   $\sum_{d \in D^c} a_d \text{lin}\{C_g^d(f)\} = 0$ . We will then show that this must be true for  $c = M$ .

In order to see this claim it will again be more useful to consider the *non-linearized* version of (70); in any dimension  $N \geq n$ :

$$E_{g^N}(f) = \sum_{u \in U_\sigma^\mu} a_u C_{g^N}^{u,+}(f, \phi) + \sum_{d \in D} a_d C_{g^N}^d(f, \phi) - \sum_{p \in P} a_p C_{g^N}^p(f, \phi) = 0 \quad (92)$$

where the complete contractions  $C_{g^N}^p(f, \phi)$  are in the form (56) and have at least  $\sigma + 2$  factors.

Now, we take the  $\text{Var}_Y^M[E_{g^N}(f, \phi)]$  in the above equation, so we will have:

$$\text{Im}_Y^M[E_{g^N}(f, \phi)] = 0 \quad (93)$$

We want to describe the left hand side of the above. For each  $d \in D^M$  let us denote by  $C_{g^N}^d(f, \phi, Y)$  the tensor field that arises from  $C_{g^N}^d(f, \phi)$  by replacing each of the  $M$  factors  $\nabla_{r_1 \dots r_p}^{(p)} S$  by a factor  $-2\nabla_{r_1 \dots r_p}^{(p)} \Delta Y$ . By virtue of the transformation laws (22) and (21), and by applying the same argument as for the previous case we derive that:

$$\sum_{d \in D^M} a_d C_{g^N}^d(f, \phi, Y) = 0 \quad (94)$$

modulo complete contractions of greater length, and moreover the above holds formally.

We then denote by  $C_{g^N}^{d,i_1 \dots i_{\mu-M-1}}(f, \phi, Y)$  the tensor field that arises by making each of the internal contractions in  $C_g^d(f, \phi, Y)$  into a free index.

By the same argument as for the previous Lemma, we derive an equation:

$$\sum_{d \in D^M} a_d \text{lin}\{C_{g^N}^{d,i_1 \dots i_{\mu-M-1}}(f, \phi, Y) \nabla_{i_1} v \dots \nabla_{i_{\mu-M-1}} v\} = 0 \quad (95)$$

and the above holds formally. Now, we observe that in each factor  $\nabla^{(A)} Y$  ( $A \geq 1$ ), the last index is contracting against a factor  $\nabla v$ . Thus, it is not difficult to derive that we can make the above linear combination formally zero *without* permuting the last index in each of the  $M$  factors  $\nabla^{(p)} Y$ . Now, we define an operation  $Op\{\dots\}$  that acts on the complete contractions above by replacing each expression  $\nabla_{r_1 \dots r_p}^{(p)} Y (\nabla)^{r_p} v$  (i.e. we pick out a factor  $\nabla_{r_1 \dots r_p}^{(p)} Y$  and also the factor  $\nabla v$  that is contracting against its last index) by an expression  $\frac{1}{2} \nabla_{r_1 \dots r_{p-1}}^{(p-1)} S$ . Furthermore, we replace each remaining factor  $\nabla v$  by an internal contraction—i.e. we erase each factor  $\nabla_r v$  that is contracting against some factor  $T$  ( $T = \nabla^{(p)} f$  or  $T = \nabla^{(m)} R$  or  $T = \nabla^{(y)} S$ ) and then we replace  $T$  by  $\nabla^r T$ . Then, since (95) holds formally and without permuting the last index  $r_p$  in each factor  $\nabla_{r_1 \dots r_p}^{(p)} Y$ , just by repeating these permutations we derive that:

$$\sum_{d \in D^m} a_d Op[\text{lin}\{C_g^{d,i_1 \dots i_{\mu-1-M}}(f, Y) \nabla_{i_1} v \dots \nabla_{i_{\mu-1-M}} v\}] = 0 \quad (96)$$

This is precisely our desired conclusion.  $\square$

## 8 Sketch of the proof of Theorem 4.

As mentioned, this proof closely follows the methods developed in [6]. All facts that we claim without proof come from [6]. We briefly present the argument that appears in [3].

The first step is to express the Riemannian operator  $L_g(f)$  as a polynomial in the components of the ambient curvature  $\tilde{R}$  and its covariant derivatives  $\tilde{\nabla}^{(m)} \tilde{R}$ , and in the harmonic extension  $\tilde{u}_w$  and its covariant derivatives  $\tilde{\nabla}^{(p)} \tilde{u}_w$ . This can be done by using Graham's *conformal normal scale*: We have that for any  $x_0 \in M$  we can pick a metric  $g_1 \in [g]$  so that  $\nabla_{(r_1 \dots r_p)}^{(p)} Ric_{r_{p+1} r_{p+2}}(x_0) = 0$  for  $p \leq N$ , where we are free to pick  $N$  as large as we like.

Now, since  $L_g(f)$  is assumed to be conformally invariant, it would suffice to show that  $L_{g_1}(f)$  can be written as a Weyl operator (the conformal invariance will then imply that  $L_g(f)$  can also be written as a Weyl operator). Therefore we consider  $L_{g_1}(f)$  and perform the ambient metric construction for  $(M, g_1)$ , as explained in subsection 4.1: We locally embed the manifold  $(M, g_1)$  into the ambient pseudo-Riemannian  $(\tilde{G}_1, \tilde{g}_1)$ , mapping the point  $x_0 \in M$  to  $x_* = (1, x_0, 0) \in \tilde{G}$ , and we consider the extension of the density  $f_w$  to a  $w$ -homogeneous harmonic function  $\tilde{u}_w$  on  $(\tilde{G}_1, \tilde{g}_1)$ . Recall that since  $(w + \frac{n}{2}) \notin \mathbb{Z}_+$ ,

this extension is well-defined to any order. Therefore, we obtain that there exists a fixed polynomial  $\Pi(\{\tilde{\nabla}^{(m)}\tilde{R}\}, \{\tilde{\nabla}^{(p)}\tilde{u}_w\})$  in the components of the tensors  $\tilde{\nabla}^{(m)}\tilde{R}$  and  $\tilde{\nabla}^{(p)}\tilde{u}_w$  so that:

$$L_{g_1}(f) = \Pi(\{\tilde{\nabla}^{(m)}\tilde{R}\}, \{\tilde{\nabla}^{(p)}\tilde{u}_w\}) \quad (97)$$

Now, consider any conformal transformation  $\psi : (M, g_1) \rightarrow (M, g_2)$  with  $\psi(x_0) = x_0$ ,  $\nabla\psi(x_0) = m_j^i \in O(g(x_0))$ , for which  $\psi^*g_2 = e^{2\phi}g_1$ . Now, perform the ambient metric construction  $(\tilde{G}_2, \tilde{g}_2)$  for  $g_2$ , mapping  $x_0$  to the point  $x_* = (1, x_0, 0) \in \tilde{G}_2$ . As discussed in subsection 4.1, there then exists an isometry (mod  $O(\rho^\infty)$ ),  $\Phi : \tilde{G}_2 \rightarrow \tilde{G}_1$ , for which  $\Phi(1, x_0, 0) = (e^{x_0}, x_0, 0)$  and moreover  $\nabla\Phi(1, x_0, 0)$  is given by the matrix:

$$\begin{bmatrix} \lambda & \omega_i & t \\ 0 & m_j^i & s^i \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \quad (98)$$

where we set  $\lambda(x) = e^{\phi(x)}$  and then  $\lambda = \lambda(x_0)$ ,  $\omega_i = \nabla_i\lambda(x_0)$ ,  $m_j^i \in O(g(x_0))$ ,  $t = -\frac{1}{2\lambda}\omega_j\omega^j$  (note that  $t$  is *not* the coordinate function from above-we are just following the notations from [6]),  $s^i = -\frac{1}{\lambda}m^{ij}\omega_j$ . Denote this matrix by  $A$ . Thus, under a conformal re-scaling above, we have that the components of the tensors  $\tilde{\nabla}_{a_1\dots a_m}^{(m)}\tilde{R}_{a_{m+1}\dots a_{m+4}}$  and  $\tilde{\nabla}_{b_1\dots b_p}^{(p)}\tilde{u}_w$  for  $\tilde{g}_2$  at  $(1, x_0, 0)$  arise from the components of those tensors of  $\tilde{g}_1$  at  $(e^{\phi(x_0)}, x_0, 0)$  by multiplying the vectors  $X_i$  by the matrix  $A$ . Denote the polynomial that is thus obtained by  $\Pi(\{A \cdot \tilde{\nabla}^{(m)}\tilde{R}\}, \{A \cdot \tilde{\nabla}^{(p)}\tilde{u}_w\})$ .

On the other hand, the conformal invariance of  $L_{g_1}(f)$  guarantees that:

$$\begin{aligned} \Pi(\{A \cdot \tilde{\nabla}^{(m)}\tilde{R}\}, \{A \cdot \tilde{\nabla}^{(p)}\tilde{u}_w\})(e^{\phi(x_0)}, x_0, 0) &= [L_{g_2}(e^{w\phi(x)}f)](x_0) = \\ [e^{(\kappa \cdot w - K)\phi(x)}L_{g_1}(f)](x_0) &= e^{(\kappa \cdot w - K)\phi(x)}\Pi(\{\tilde{\nabla}^{(m)}\tilde{R}\}, \{\tilde{\nabla}^{(p)}\tilde{u}_w\}) \end{aligned} \quad (99)$$

Thus, the conformal invariance of  $L_g(f)$  translates into an invariance of the polynomial  $\Pi$  (up to multiplying by a power of  $\lambda$ ) under the action of the group  $P$  of matrices in the form (98). We will call this property  $P$ -invariance.

Now, it is more convenient to work at the linearized level of modules.

We consider the vector space  $W = \mathbb{R}^{n+2}$  and we denote by  $X^I = (X^0, X^1, \dots, X^n, X^\infty)$  any point in  $W$ . We recall the metric  $\tilde{g}(x_*)$  from the subsection 4.1, which now defines a quadratic form  $\tilde{g}$  on  $W$ :  $\sum_{1 \leq I, J \leq n} g_{IJ}X^IX^J + 2X^0X^\infty$ . We denote by  $Q$  the light cone in  $W$ , with respect to this quadratic form. We consider the null vector  $e_0 = (1, 0, \dots, 0)^t \in W$ . Now, if we denote by  $G$  the identity-connected component of  $O(\tilde{g})$  then the group of matrices  $P$  in the form (98) above stands for the parabolic subgroup of  $G$ :

$$P = \{p \in G : pe_0 = \lambda e_0, \lambda > 0\}.$$

We will now be considering jets (to infinite order), at  $e_0$ , of homogeneous functions and tensor fields. Homogeneity here refers to the usual dilations of the space  $W = \mathbb{R}^{n+2}$ . We denote by  $E(a)$  the space of jets of functions  $u_a$ , homogeneous of degree  $a$  and by  $E^{IJ\dots M}(a)$  (there are  $m$  indices  $I, J, \dots, L$ ) the space of jets of functions of homogeneity  $a$  that take values in the space  $W \otimes \dots \otimes W$  (we are tensoring  $m$  times). For example  $E^{IJK}$  is just a convenient way of recording that the jets take values in  $W \otimes W \otimes W$ . Moreover,  $E_I{}^{JK}$  is the space of jets of functions that take values in  $W^* \otimes W \otimes W$ . We denote by  $H(a)$  the space of jets at  $e_0$  of homogeneous *harmonic* functions  $u_a$ . *Harmonic* here means with respect to the operator  $\Delta = \tilde{g}^{IJ}\partial_{IJ}^2$ .

We define the space  $K$  of *jets of linearized ambient curvature tensors* around a point  $e_0 \in \mathbb{R}^{n+2}$  to be the set of jets to infinite order of 4-tensor fields  $\rho_{IJKL}$ , with homogeneity  $-2$ , that satisfy the identities:

$$\rho_{[IJ]KL} = 0, X^L \rho_{IJKL} = 0, \partial_{[H} \rho_{IJ]KL} = 0, \rho_{[IJK]L} = 0, \rho_{IJ[KL]} = 0$$

and moreover each tensor  $\partial_{A\dots C}^{(m)} \rho_{IJKL}$  is trace-free with respect to the quadratic form  $\tilde{g}$ . Here  $[\dots]$  stands for summation over all cyclic permutations of the indices inside the brackets.

We then define the function *Eval*, that evaluates each such jet at  $e_0$ . Note that  $X = (X^I)$  is an element of  $E^I(1)$ . We then denote  $e^I = \text{Eval}(X^I)$ . We recall that coordinate differentiation defines a  $P$ -invariant map:

$$\partial_I : E^{JK\dots M}(s) \longrightarrow E_I^{JK\dots M}(s-1)$$

Now, let us define  $\text{lin}\Pi(H(a), K)$  to be the polynomial in  $H(a) \oplus K$  that arises from  $\Pi(\tilde{u}_a, \tilde{R})$  by replacing each factor  $\tilde{\nabla}_{AB\dots D}^{(m)} \tilde{R}_{IJKL}$  by a factor  $\partial_{AB\dots D}^{(m)} \rho_{IJKL}$  and each factor  $\tilde{\nabla}_{AB\dots D}^r \tilde{u}_a$  by a factor  $\partial_{AB\dots D}^r u_a, u_a \in H(a)$ . For each  $p \in P$ , we define  $\text{plin}\Pi(H(a), K)$  to stand for the complete contraction that arises from  $\text{lin}\Pi(H(a), K)$  in the following way: Let  $p = (q_j^i)$  be in the form (98) and  $\lambda$  be the top-left component of the matrix  $p$ . Denote the index  $\infty$  by  $n+1$  for convenience. Then  $\text{plin}\Pi(H(a), K)$  arises from  $\text{lin}\Pi(H(a), K)$  by substituting each factor  $\partial_{AB\dots D}^{(m)} \rho_{IJKL}$  by a factor  $\lambda^2 \sum_{A', B', \dots, D', I', J', K', L'=0}^{n+1} \partial_{A' B' \dots D'}^{(m)} \rho_{I' J' K' L'} q_{A'}^{A'} \dots q_{L'}^{L'}$ . We also replace each factor  $\partial_{AB\dots D}^r u_a$  by a factor  $\lambda^a \sum_{A', B', \dots, D'=0}^{n+1} \partial_{A' B' \dots D'}^r u_a q_{A'}^{A'} \dots q_{D'}^{D'}$  and each factor  $\tilde{g}^{IJ}$  by  $\lambda^{-2} \sum_{I', J'=0}^{n+1} \tilde{g}^{I' J'} q_{I'}^I q_{J'}^J$ .

We recall the following fundamental fact, that follows from [10]:

**Proposition 8** *In the notation above, let us suppose that we can show that if  $\text{lin}\Pi(H(a), K)$  is  $P$ -invariant,  $\text{lin}\Pi(H(a), K) : H(a) \oplus K \longrightarrow E(b)$ , then  $\text{lin}\Pi(H(a), K)$  can be written as:*

$$\text{lin}\Pi(H(a), K) = \sum_{h \in H} a_h C^h(H(a), K)$$

where each  $C^h(H(a), K)$  is a complete contraction (in  $\tilde{g}$ ), in the form:

$$\begin{aligned} & \text{contr}_{\tilde{g}}(\partial_{AB\dots D}^{(m_1)} \rho_{IJKL} \otimes \cdots \otimes \partial_{A'B'\dots D'}^{(m_s)} \rho_{I'J'K'L'} \otimes \\ & \partial_{FG\dots H}^{(p_1)} u_a \otimes \cdots \otimes \partial_{F'G'\dots H'}^{(p_o)} u_a) \end{aligned} \quad (100)$$

with  $\sum_{i=1}^s (m_i + 2) + \sum_{i=1}^o p_i = K$  and  $o = q$ .

It then follows that  $\Pi(\tilde{R}, \tilde{u}_a)$  can be written in the form:

$$\Pi(\tilde{R}, \tilde{u}_a) = \sum_{h \in H'} a_h \tilde{C}_{\tilde{g}}^h(\tilde{u}_a)$$

where each  $\tilde{C}_{\tilde{g}}^h(\tilde{u}_a)$  is in the form (11) with  $\sum_{i=1}^s (m_i + 2) + \sum_{i=1}^o p_i = K$  and  $o = q$ .

In view of the above, the rest of this section will focus on proving the hypothesis of the above Proposition. It will prove useful to establish two isomorphisms between the space of jets at  $e_0$  of homogeneous harmonic functions and linearized ambient curvatures, and the space of two lists of tensors, which we will denote by  $H_{list}(a), K_{list}$ . We use Propositions 1.2 and 4.1 from [6] to establish these two isomorphisms.

Proposition 1.2 in [6] implies that if  $a \notin \mathbb{Z}_+$  (which is the case here, since we are assuming  $a = w \notin \mathbb{Z}_+$ ) the space  $H(a)$  of jets at  $e_0$  of  $a$ -homogeneous harmonic functions, is isomorphic to the  $P$ -module of lists:

$$H_{list}(a) = \{T^0, T^1, T^2, \dots\}, T^l \in \odot_0^l W^* \otimes \sigma_{a-l} \quad (101)$$

where  $e^I(T^{l+1})_{IJ\dots M} = (a-l)(T^l)_{J\dots M}$ . Here  $\sigma_q$  is the 1-dimensional representation of  $P$  where the element of  $P$  in (98) is mapped to  $\lambda^{-q}$ . Also,  $\odot^l$  stands for the symmetrized  $l$ -tensor product and  $\odot_0^l$  stands for the trace-free part of the symmetrized  $l$ -tensor product. These conditions reflect the fact that we are dealing with densities of weight  $a$ , the trace-free restriction reflects the harmonicity of  $u_a$  and the restriction on the contraction against  $e^I$  reflects the Euler homogeneity relations.

We also recall the Proposition 4.1 in [6] which shows that that  $K$  is isomorphic to a certain  $P$ -module of lists of tensors, denote it by  $K_{list}$ , with symmetries and anti-symmetries that model the usual properties of the curvature and its covariant derivatives. We refer the reader to that paper. To avoid confusion, we will denote the tensors  $T^{(l)}$  in Proposition 4.1 in [6] by  $Q^l$ .

Now, we will revert from thinking of the vector space of jets to thinking of  $P$ -modules of lists of tensors. We thus study a polynomial  $\Pi(H_{list}(a), K_{list})$  in elements of the lists  $H_{list}(a), K_{list}$ .

Now, by Weyl's classical invariant theory and using the fact that  $O(g(x_0)) \subset P$ , we have that  $\Pi(H_{list}(a), K_{list})$  can be written as:

$$\Pi(H_{list}(a), K_{list}) = \Pi_{even}(H_{list}(a), K_{list})$$

where  $\Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$  is a linear combination of complete contractions in the form:

$$\text{contr}_g(T^{k_1} \otimes \dots \otimes T^{k_s} \otimes Q^{l_1} \otimes \dots \otimes Q^{l_r}) \quad (102)$$

the contractions are with respect to the metric  $g$ .

This reflects the fact that a conformally invariant differential operator is still a Riemannian operator.

Now, recall from [6] the notion of a weak Weyl invariant: Assume that  $C$ ,

$$C : H_{\text{list}}(a) \oplus K_{\text{list}} \longrightarrow \odot_0^{(m)} W \otimes \sigma_{b+m}$$

is a linear combination of partial contractions (with respect to the metric  $\tilde{g}$ ) of the tensors  $Q^l$ ,  $T^k$ ,  $e^I$  (which we denote by  $e$ , for short). If  $C$  is of the form:

$$C = e \otimes \dots \otimes e \otimes I$$

where the factor  $e$  is tensored  $m$  times and  $I$  is a polynomial,

$$I : H_{\text{list}}(a) \oplus K_{\text{list}} \longrightarrow \sigma_b$$

then we observe that  $I$  must be  $P$ -invariant. (This follows because  $Pe = \lambda e$ ). We will then call  $I$  a *weak Weyl invariant*.

So the  $m$ -tensor  $C^{AB\dots D}$  has  $C^{0\dots 0} = I$  and all the other components vanish. Equivalently, the tensor  $C_{AB\dots}$  has  $C_{\infty\dots\infty} = I$  and all the other components vanish. We then claim:

**Proposition 9** *Our polynomial  $\Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$  can be written as a weak Weyl invariant.*

*Proof:* Using the quadratic form  $\tilde{g}$ :

$$\tilde{g}_{IJ} X^I X^J = g_{ij} X^i X^j + 2X^0 X^\infty,$$

we see that  $\Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$  can be written as linear combination of complete contractions (in  $\tilde{g}$ ) of the tensors  $T_{AB\dots D\infty\dots\infty}^l$ ,  $Q_{IJKL,AB\dots D\infty\dots\infty}^r$ ,  $Q_{IJK\infty,AB\dots D\infty\dots\infty}^r$  and  $Q_{I\infty K\infty,AB\dots D\infty\dots\infty}^r$  (where the indices  $A, B, \dots, D, I, J, K, L$  take the values  $0, 1, \dots, n, \infty$ ).

By juxtaposing  $e_I = (0, \dots, 0, 1)$  if necessary, we may assume that the number of  $\infty$  indices is equal to  $m$  in all the terms in that linear combination.

Then, we make all the  $\infty$ 's into free indices  $X, Y, \dots, Z$  and so we have an  $m$ -tensor  $F_{XY\dots Z}$ . We then take the symmetric and trace-free part of  $F_{XY\dots Z}$ , say  $C_{XY\dots Z}$ . Because  $\tilde{g}_{\infty\infty} = 0$ , we still have that  $C_{\infty\dots\infty} = \Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$ . Equivalently, raising indices we have that  $C^{0\dots 0} = \Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$ .

We then consider  $D_{XY\dots Z} = C_{XY\dots Z} - e \otimes \dots \otimes e \otimes \Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}})$ . All we need to show is that  $D_{XY\dots Z}$  vanishes. In other words, this is a linear algebra problem: Given the quadratic form  $\tilde{g}$  and an  $m$ -form  $C^{XY\dots Z}$  that is  $P$ -invariant and symmetric and totally trace-free so that  $C^{0\dots 0} = 0$ , then show

that  $C^{XY\dots Z} = 0$ . But this is shown in Proposition 2.1 in [6] (we only need the even case here).  $\square$

What remains to be done is to show that the term  $I$  in  $\Pi_{\text{even}}(H_{\text{list}}(a), K_{\text{list}}) = e \otimes \dots \otimes e \otimes I$  is a Weyl invariant. But this part of the argument exactly follows the ideas in [6]:

We now revert to thinking of the modules  $H(a)$  and  $K$  as jets of homogeneous harmonic functions and linearized curvature tensors around  $e_0$ , rather than just lists of tensors.

Let  $F^{AB\dots G}(q+m)$  denote the space of jets at  $e_0$  of restrictions of  $m$ -tensors (with homogeneity  $q+m$ ) to the light cone  $Q$ .

We now use the differential operator  $D_I$  from [6] that acts on elements of  $F^{AB\dots E}(s)$  and maps them into  $F_I^{AB\dots E}(s-1)$ . We pick any  $f^{AB\dots E} \in F^{AB\dots E}(s)$  and arbitrarily extend it off of  $Q$  to an element of  $E^{AB\dots E}(s)$ . We then define:

$$D_I f^{AB\dots E} = (\partial_I - \frac{X_I \Delta f^{AB\dots E}}{(n+2s-2)})|_Q \quad (103)$$

It follows that this operation is independent of the extension off of  $Q$  (see [6]). Moreover, we observe that for any  $f \in F(s)$ ,  $n+2s \neq 0$ , we have:

$$D_I(X^I f) = \frac{(n+2s+2)(n+s)}{(n+2s)} f \quad (104)$$

Repeatedly applying this operator as in [6], we then conclude that any weak Weyl invariant with  $b = (\kappa \cdot w - K) + m$ ,  $m \in \mathbb{Z}_+$ , with  $w$  subject to the restrictions of Theorem 4, will be a Weyl invariant. We have proven our Theorem 4. (Note: The restriction  $(w + \frac{n}{2}) \notin \mathbb{Z}$  ensures that the operation is well-defined; The last two restrictions ensure that the constant on the right hand side of (104) is not zero).  $\square$

## 9 Appendix: Proof of Lemma 2.

We first prove our claim for complete contractions in the form (1). Let us denote by  $\omega$  the number of factors of  $C_g(f)$ . It is straightforward to observe that if  $C_g(f)$  satisfies the extra restrictions then all contractions  $C_g^k(f)$  with  $\omega$  factors (in the notation of Lemma 2) will also satisfy the extra restrictions. Thus, we may restrict attention to the contractions  $C_g^k(f)$  with  $\omega+1$  factors.

By definition, such contractions can arise in this setting only by applying the curvature identity. But then our claim follows because whenever we apply the curvature identity to a factor  $F_h = \nabla^{(p)} f$  with  $\beta[F_h] = \chi$ , or a factor  $F_h = \nabla^{(m)} R$  with  $\gamma[F_h] = \chi$ , we will obtain a linear combination of partial contractions in the forms  $\nabla^{(m')} R \otimes \nabla^{(p')} f$  or  $\nabla^{(m')} R \otimes \nabla^{(m'')} R$ , respectively. Denote these factors by  $F'_h, F''_h$ . It follows that we will then have  $\beta[F''_h] \leq \chi-2$  and  $\gamma[F'_h] \leq \chi$ . Thus, since we started off with a complete contraction that satisfied the extra restrictions, we obtain a complete contraction that satisfies

the extra restrictions.

Now we prove our claim for complete contractions in the form (24). In that setting, our claim is obvious if  $C_g^h(f)$  arises from  $C_g(f)$  by switching two derivative indices, or by applying (23) to a factor  $\nabla^{(p)}P$  or by applying a fake second Bianchi identity provided we *do not* increase the number of factors and also *do not* increase the number of internal contractions. The remaining cases in which a contraction  $C_g^k(f)$  may arise from  $C_g(f)$  are as follows:

Firstly,  $C_g^k(f)$  may arise by applying the Ricci identity to two derivative indices in the same factor. Secondly by replacing a factor  $\nabla^{(m)}W$  by one of the summands  $\sum(\nabla^s W_{sjkl} \otimes g)$ ,  $\sum(\nabla^{ik} W_{ijkl} \otimes g)$  in the right hand sides of (51), (52). Thirdly, by the expressions  $Q(R)$  in the right hand sides of (52), (53).

Now, in the first case our claim follows by the same argument as for  $C_g(f)$  being in the form (1). For the second case we observe that whenever we apply (51) or (52) to a factor  $F_h$  and we introduce a correction term of the form  $F'_h = (\nabla^s W_{srtj} \otimes g)$  or of the form  $F'_h = (\nabla^{st} W_{srtj} \otimes g)$  respectively, we obtain a complete contraction with at least  $\mu + 1$  internal contractions, and where the number of derivatives on each factor remains unaltered. Thus, we only have to check that the extra restrictions continue to hold whenever they are applicable. To see this, we observe by definition that if we had  $\gamma[F_h] = \chi$  before we applied (51) or (52), we then have  $\gamma[F'_h] = \chi - 1$ . Furthermore, if we create an internal contraction in some other factor  $T$  by virtue of the un-contracted metric tensor, we again decrease the quantities  $\gamma[T]$ ,  $\beta[T]$ . All the other factors  $T'$  will still have the same numbers  $\beta[T']$ ,  $\gamma[T']$ , so we derive our claim.

The third case is when  $C_g^h(f)$  arises from  $C_g(f)$  when we apply one of the equations (52), (53) to a factor  $F_h = \nabla^{(m)}W$  and bring out the correction terms in  $Q(R)$ . Suppose that  $C_g^k(f)$  arises by applying (52) or (53) to a factor  $F_h = \nabla^{(m)}W$ . We only have to check that  $F_h$  is then being replaced by a partial contraction  $\nabla^{(m')}R \otimes \nabla^{(m'')}R$  for which  $\gamma[\nabla^{(m')}R] < \frac{n}{2}$ ,  $\gamma[\nabla^{(m'')}R] < \frac{n}{2}$ , whenever this extra restriction is applicable. Now, for this we see that  $Q(R)$  is a sum of partial contractions in the form  $R \otimes R$ . But each such curvature expression has  $\gamma[R] \leq 2$ , whereas the expression  $F_h$  in the left hand sides of (52), (53) have  $\gamma[F_h] = 3$ . Furthermore, we observe that  $m' + m'' = m - 2$  and that each derivative index  $r$  in  $F_h$  that does not belong to an internal contraction in  $F_h$  will belong to only one of the factors  $\nabla^{(m')}R$ ,  $\nabla^{(m'')}R$ , while each internal contraction in  $F_h$  will either give rise to an internal contraction in one of the factors  $\nabla^{(m')}R$ ,  $\nabla^{(m'')}R$ , or it will give rise to two indices  $a, b$  in  $\nabla^{(m')}R$ ,  $\nabla^{(m'')}R$  respectively that will contract against each other. Thus, since  $F_h$  satisfied  $\gamma[F_h] < \frac{n}{2}$ , we obtain that  $\gamma[\nabla^{(m')}R] < \frac{n}{2}$ ,  $\gamma[\nabla^{(m'')}R] < \frac{n}{2}$ .  $\square$

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